Using Mixture Latent Markov Models for Analyzing Change with Longitudinal Data

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Abstract

Mixture latent Markov (MLM) models are latent class models containing both time-constant and time-varying discrete latent variables. We introduce and illustrate the utility of latent Markov models in 3 real world examples using the new GUI in Latent GOLD® 5.0.

MLM models often fit longitudinal data better than comparable mixture latent growth models because of the inclusion of transition probabilities as model parameters which directly account for first-order autocorrelation in the data. Lack of fit can be localized (e.g., first-order, second-order autocorrelation) using new longitudinal bivariate residuals (L-BVRs) to suggest model modification.

Outline of Presentation

- Introduction to latent Markov (LM) and mixture latent Markov (MLM) Models*

- Example 1: LM model with time-homogeneous transition probabilities
- Example 2: MLM model with time-heterogeneous transitions
- Example 3: MLM model with time-heterogeneous transitions and covariates

In each of these examples latent Markov outperformed the corresponding latent growth model, the largest improvement being in the Lag1 BVRs (i.e., first-order autocorrelation).

* (Mixture) *latent* Markov models are also known as (mixture) *hidden* Markov models

Latent (Hidden) Markov Models are Defined by 3 Sets of Equations

- **Initial State probs: $P(X_0 = s)$ –** categories of X are called *latent states* (rather than *latent classes*) since a person may change from one state to another over time.

Logit model may include covariates Z (e.g., AGE, SEX)

$$\log \frac{P(x_0 = s)}{P(x_0 = 1)} = \alpha_{0s}$$

- **Transition probs: $P(X_t = r | X_{t-1} = s)$**

$$\log \frac{P(x_t = r \mid x_{t-1} = s)}{P(x_t = 1 \mid x_{t-1} = s)} = \gamma_{0r} + \gamma_{1rs} + \gamma_{2rt} + \gamma_{3rst}$$

Logit may include time-varying and fixed predictors

- **Measurement model probs: $P(y_t = j | X_t = s)$**

One or more dependent variables Y, of possibly different scale types (e.g., continuous, count, dichotomous, ordinal, nominal)

$$\log \frac{P(y_{it} = \ell \mid x_t = s)}{P(y_{it} = 1 \mid x_t = s)} = \beta_{0\ell} + \beta_{1\ell s}$$

* Equations can be customized using Latent GOLD 5.0 GUI and/or syntax.

Equations:

Latent Markov (LM):

$$P(\mathbf{y}_i \mid \mathbf{z}_i) = \sum_{x_0=1}^{K} \sum_{x_1=1}^{K} \dots \sum_{x_T=1}^{K} P(x_0, x_1, \dots, x_T \mid \mathbf{z}_i) P(\mathbf{y}_i \mid x_0, x_1, \dots, x_T, \mathbf{z}_i)$$

Initial latent state & Transition sub-models

$$P(x_0, x_1, \dots, x_T \mid \mathbf{z}_i) = P(x_0 \mid \mathbf{z}_{i0}) \prod_{t=1}^{T} P(x_t \mid x_{t-1}, \mathbf{z}_{it})$$

Measurement sub-model

$$P(\mathbf{y}_i \mid x_0, x_1, \dots, x_T, \mathbf{z}_i) = \prod_{t=0}^{T} P(\mathbf{y}_{it} \mid x_t, \mathbf{z}_{it}) = \prod_{t=0}^{T} \prod_{j=1}^{J} P(y_{itj} \mid x_t, \mathbf{z}_{it})$$

Mixture Latent Markov (MLM):

$$P(\mathbf{y}_i \mid \mathbf{z}_i) = \sum_{w=1}^{L} \sum_{x_0=1}^{K} \sum_{x_1=1}^{K} \dots \sum_{x_T=1}^{K} P(w, x_0, x_1, \dots, x_T \mid \mathbf{z}_i) P(\mathbf{y}_i \mid w, x_0, x_1, \dots, x_T, \mathbf{z}_i)$$

$$P(w, x_0, x_1, \dots, x_T \mid \mathbf{z}_i) = P(w \mid \mathbf{z}_i) P(x_0 \mid w, \mathbf{z}_{i0}) \prod_{t=1}^{T} P(x_t \mid x_{t-1}, w, \mathbf{z}_{it})$$

$$P(\mathbf{y}_i \mid w, x_0, x_1, \dots, x_T, \mathbf{z}_i) = \prod_{t=0}^{T} P(\mathbf{y}_{it} \mid x_t, w, \mathbf{z}_{it}) = \prod_{t=0}^{T} \prod_{j=1}^{J} P(y_{itj} \mid x_t, w, \mathbf{z}_{it})$$

Relationship between Mixture Latent Markov and Mixture Latent Growth Models

Classification of latent class models for longitudinal research

	Model name	Transition structure	Unobserved heterogeneity	Measurement error
I	Mixture latent Markov	yes	yes	yes
II	Mixture Markov	yes	yes	no
III	Latent Markov	yes	no	yes
IV	Standard Markov*	yes	no	no
V	Mixture latent growth	no	yes	yes
VI	Mixture growth	no	yes	no
VII	Standard latent class	no	no	yes
VIII	Independence*	no	no	no

*This model is not a latent class model.

Vermunt, Tran and Magidson (2008) “Latent class models in longitudinal research”, chapter 23 in Handbook of Longitudinal Research, S. Menard Editor, Academic Press.

Longitudinal Bivariate Residuals (L-BVRs)

Longitudinal bivariate residuals quantify for each response variable Y_k how well the overall trend as well as the first- and second-order autocorrelations are predicted by the model.

BVR.Time=BVR_k(time, y_k),

BVR.Lag1 =BVRk(y_k[t-1], y_k[t])

and BVR.Lag2 =BVR_k(y_k[t-2], y_k[t])

residual autocorrelations remaining unexplained by the model.

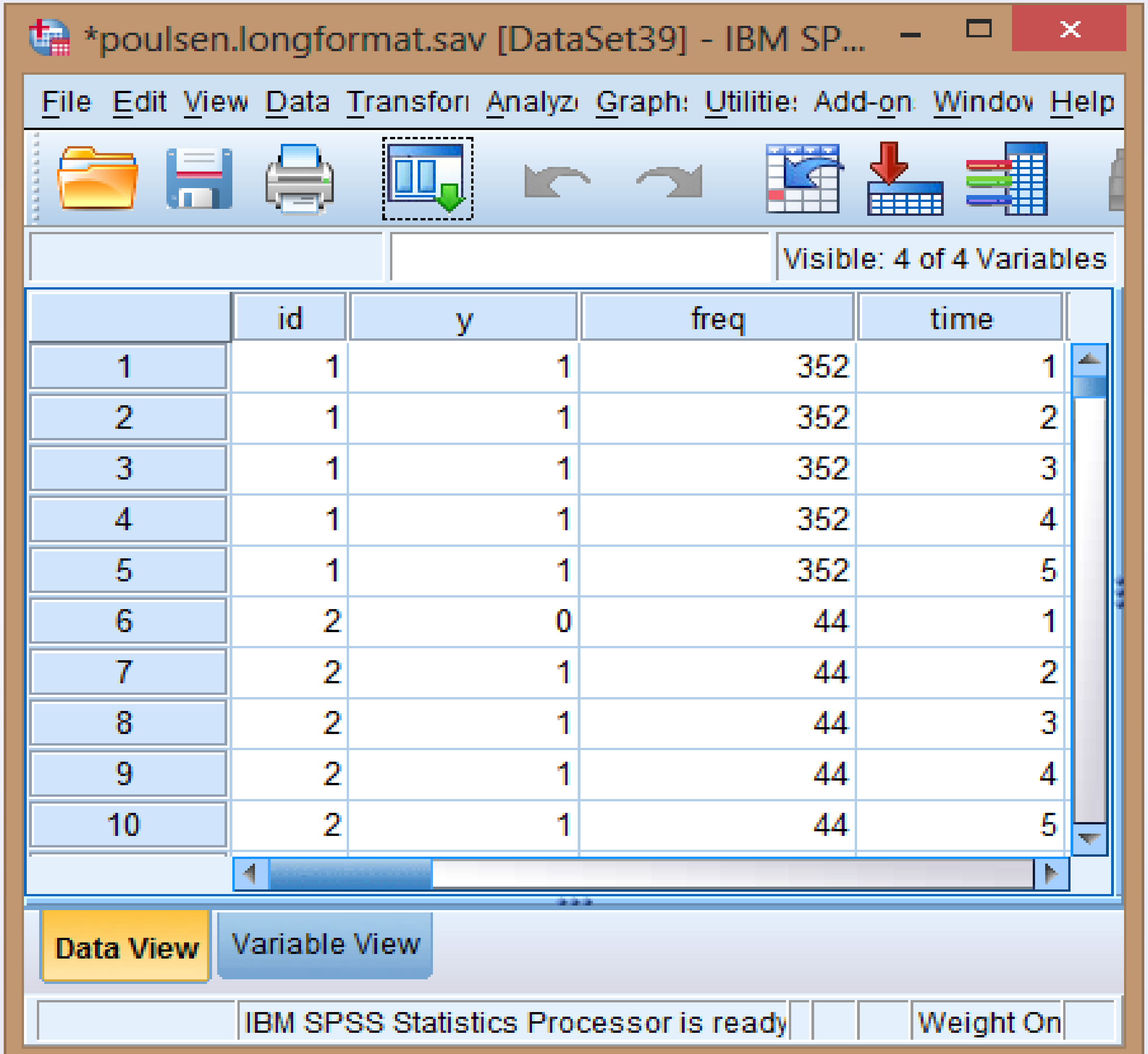
Lag1: $$BVR_k = \sum_r \sum_s \frac{(n_{krs} - m_{ksr})^2}{m_{krs}}$$

where m_{ksr} is based on $P(Y_{kit-1} = r, Y_{kit} = s)$, that is

$$m_{ksr} = \sum_i \sum_t P(Y_{kit-1} = r, Y_{kit} = s).$$

$$= P(Y_{kit-1} = r | X_{it-1} = x_{t-1}) P(Y_{kit} = s | X_{it} = x_t)$$

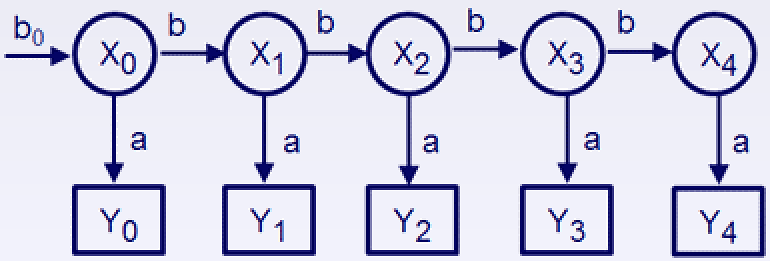
Example 1: Loyalty Data in Long File Format



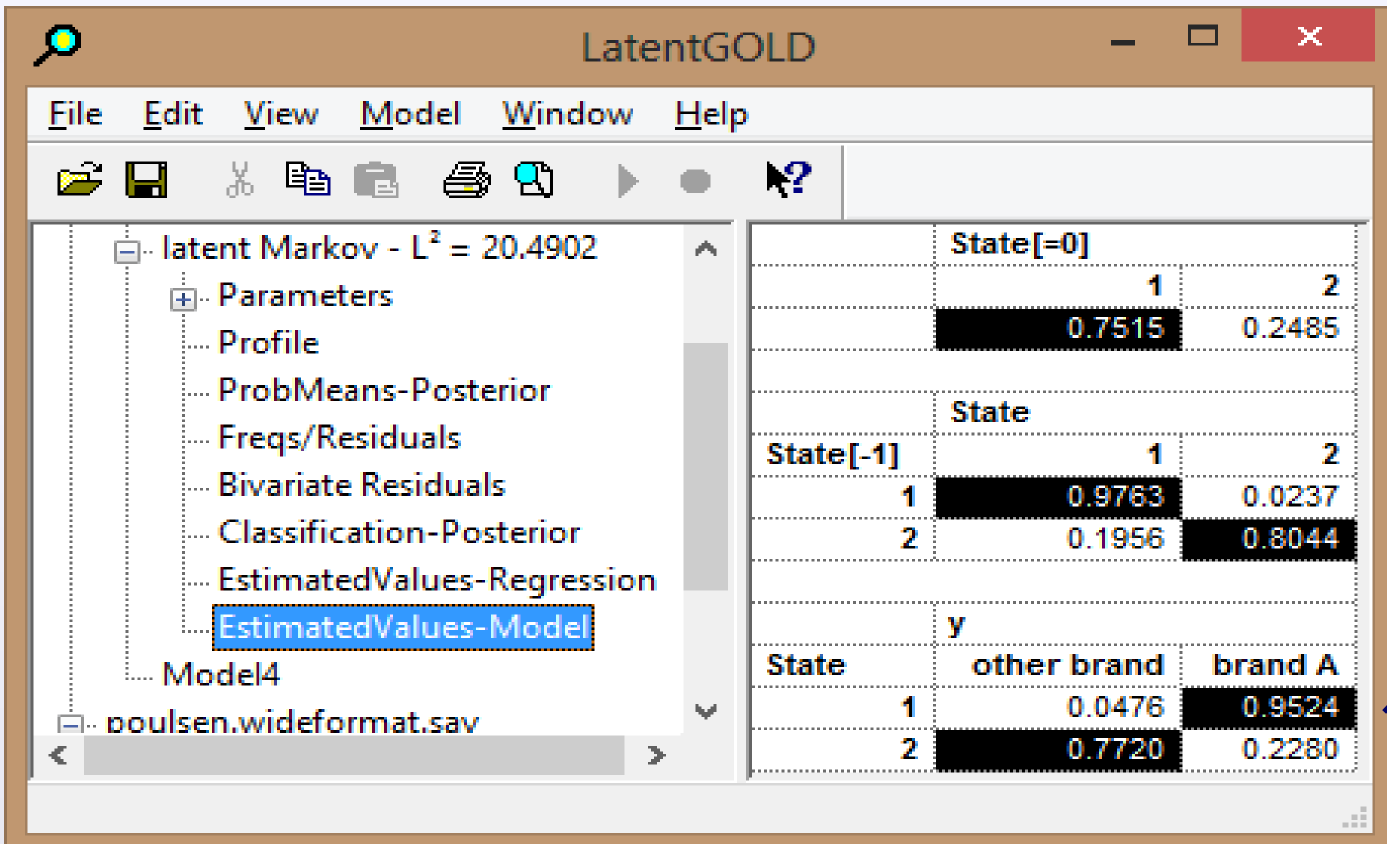
	id	y	freq	time
1	1	1	352	1
2	1	1	352	2
3	1	1	352	3
4	1	1	352	4
5	1	1	352	5
6	2	0	44	1
7	2	1	44	2
8	2	1	44	3
9	2	1	44	4
10	2	1	44	5

- N=631 respondents
- T= 5 time points
- Dichotomous Y:
  - Choose Brand A?
    - 1=Yes
    - 0=No
- $Y=(Y_1,Y_2,Y_3,Y_4,Y_5)$
- $2^5 = 32$ response patterns id=1,2,…,32
- ‘freq’ is used as a case weight

Loyalty Model with Time-homogeneous Transitions



Measurement equivalence



Initial State Probability (b_0)

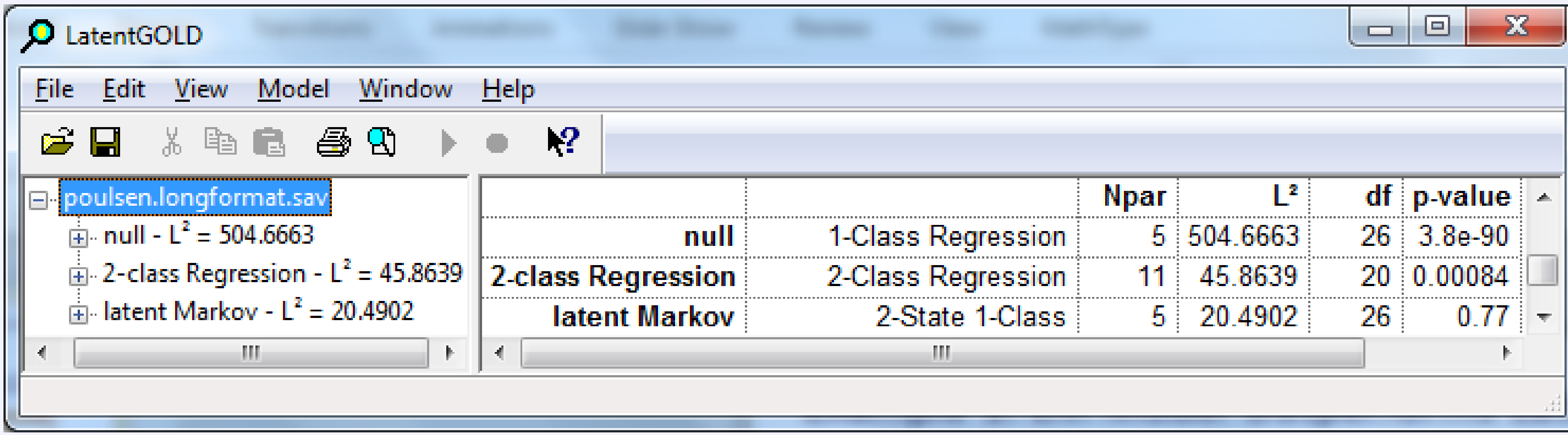
Transition Probabilities (b)

Measurement Model Probabilities (a)

Example 1: LM Model Outperforms Latent Growth Model

2-state time-homogeneous latent Markov model
fits well: (p=.77), small L-BVRs

2-class latent growth model (“2-class Regression”) is rejected (p=.0084)

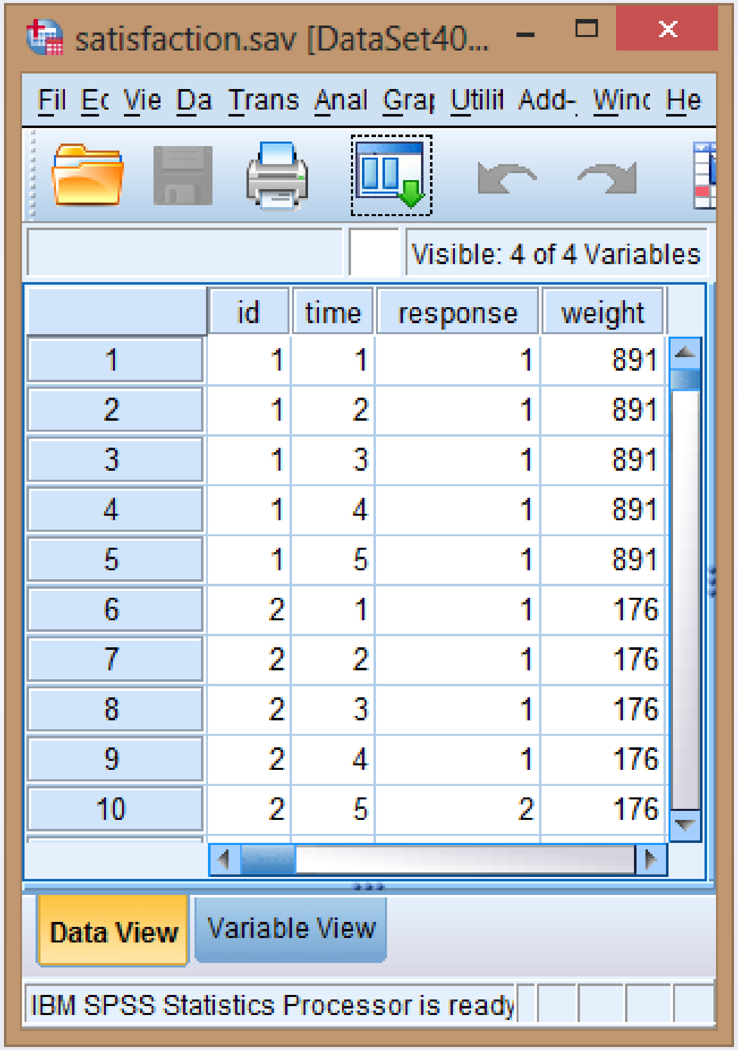


		Npar	L^2	df	p-value
null	1-Class Regression	5	504.6663	26	3.8e-90
2-class Regression	2-Class Regression	11	45.8639	20	0.00084
latent Markov	2-State 1-Class	5	20.4902	26	0.77

L-BVR	2-class Regression	LM
Time	0.000	.0785
Lag1	8.739* (p = .003)	.0115
Lag2	0.065	.3992

Lag1 BVR pinpoints problem in LC growth model as failure to explain 1st order autocorrelation

Example 2: Life Satisfaction Model with Time-Heterogeneous Transitions



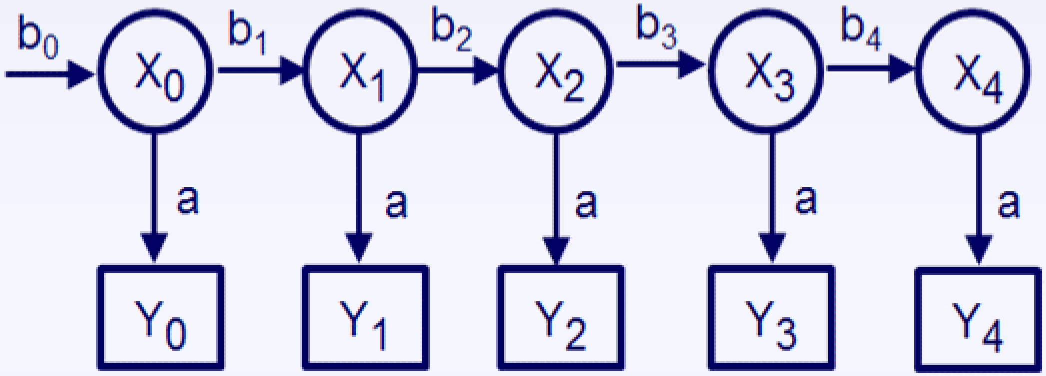
	id	time	response	weight
1	1	1	1	891
2	1	2	1	891
3	1	3	1	891
4	1	4	1	891
5	1	5	1	891
6	2	1	1	176
7	2	2	1	176
8	2	3	1	176
9	2	4	1	176
10	2	5	2	176

- N=5,147 respondents
- T= 5 time points
- Dichotomous Y:
  - Satisfied with life?
    - 1=No
    - 2=Yes
- $Y=(Y_1,Y_2,Y_3,Y_4,Y_5)$
- 2^5 = 32 response patterns id=1,2,…,32
- ‘weight’ is used as a case weight

Models:

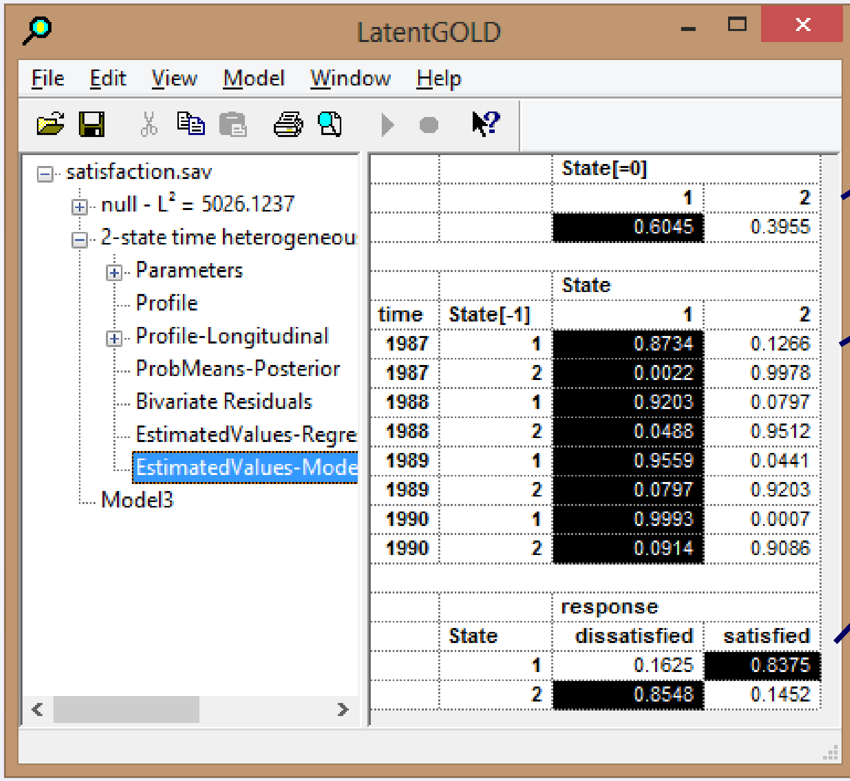
- Null
- Time heterogeneous LM
- 2-class mixture LM
- Restricted (Mover-Stayer)

Ex. 2: Time-heterogeneous Models Latent Markov Model



Separate transition probability parameters exist for each pair of adjacent time points

Example 2: Model Parameters Time-heterogeneous Model



		State[=0]	
		1	2
		0.6045	0.3955

		State	
time	State[-1]	1	2
1987	1	0.8734	0.1266
1987	2	0.0022	0.9978
1988	1	0.9203	0.0797
1988	2	0.0488	0.9512
1989	1	0.9559	0.0441
1989	2	0.0797	0.9203
1990	1	0.9993	0.0007
1990	2	0.0914	0.9086

		response	
	State	dissatisfied	satisfied
	1	0.1625	0.8375
	2	0.8548	0.1452

Initial State Probability (b_0)

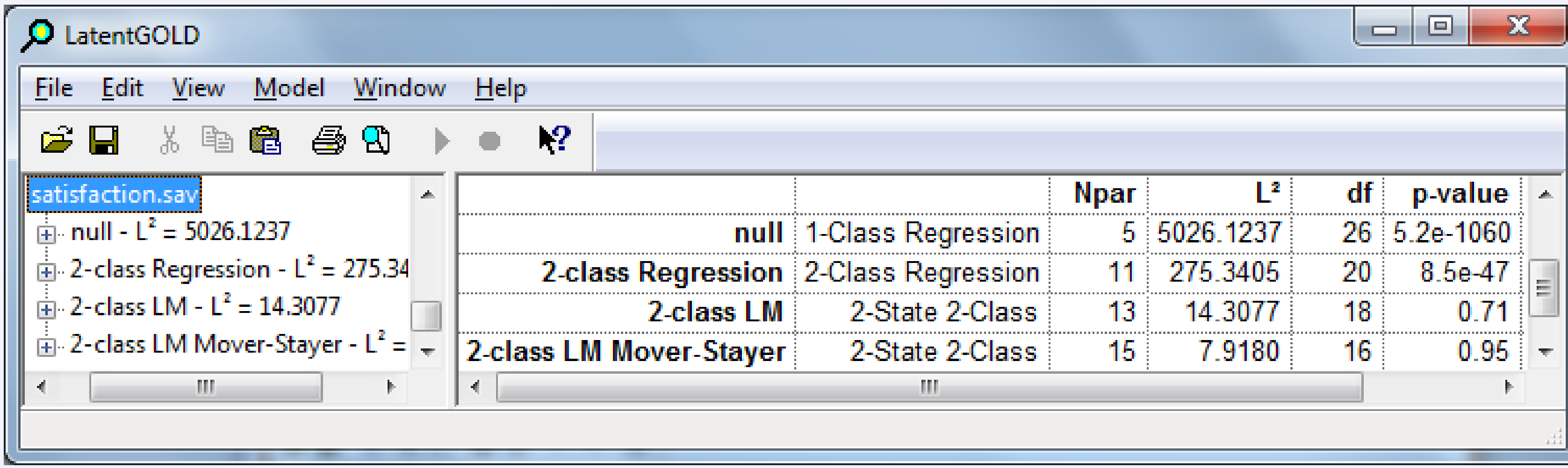
Transition Probabilities (b_t)

Measurement Model Probabilities (a)

Estimated values for 2-state LM model

Ex. 2: Mover-Stayer Time-heterogeneous Latent Markov Model

Both the unrestricted and Mover-Stayer 2-class MLM models fit well (p=.95 and .71), the BIC statistic preferring the Mover-Stayer model.

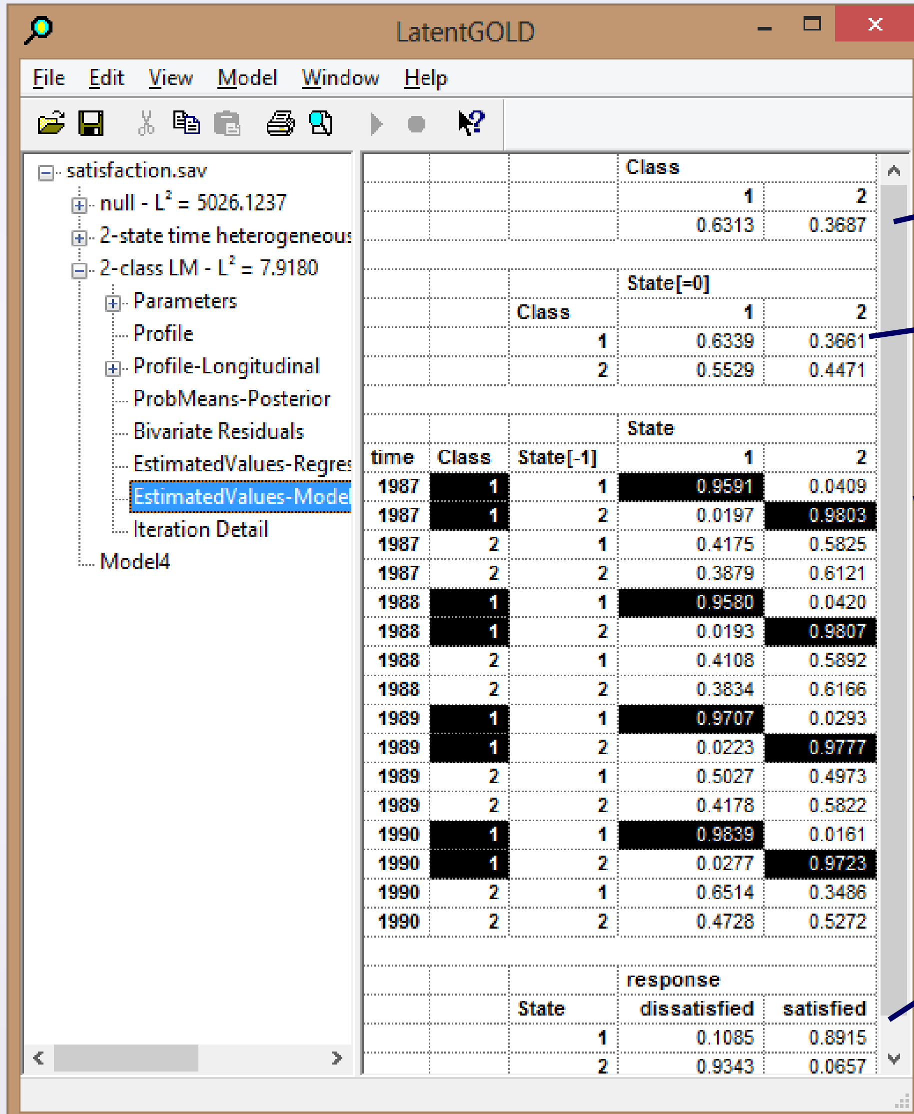


		Npar	L^2	df	p-value
null	1-Class Regression	5	5026.1237	26	5.2e-1060
2-class Regression	2-Class Regression	11	275.3405	20	8.5e-47
2-class LM	2-State 2-Class	13	14.3077	18	0.71
2-class LM Mover-Stayer	2-State 2-Class	15	7.9180	16	0.95

L-BVR	2-class Regression	2-class LM	2-class LM Mover-Stayer
Time	0.0	0.02	0.03
Lag1	55.1 (p=1.1E-13)	0.00	0.00
Lag2	1.7	0.10	1.97

Mover-Stayer Time-heterogeneous 2-class MLM Model

Estimated Values output for 2-state time-heterogeneous MLM model with 2 classes



Class Size

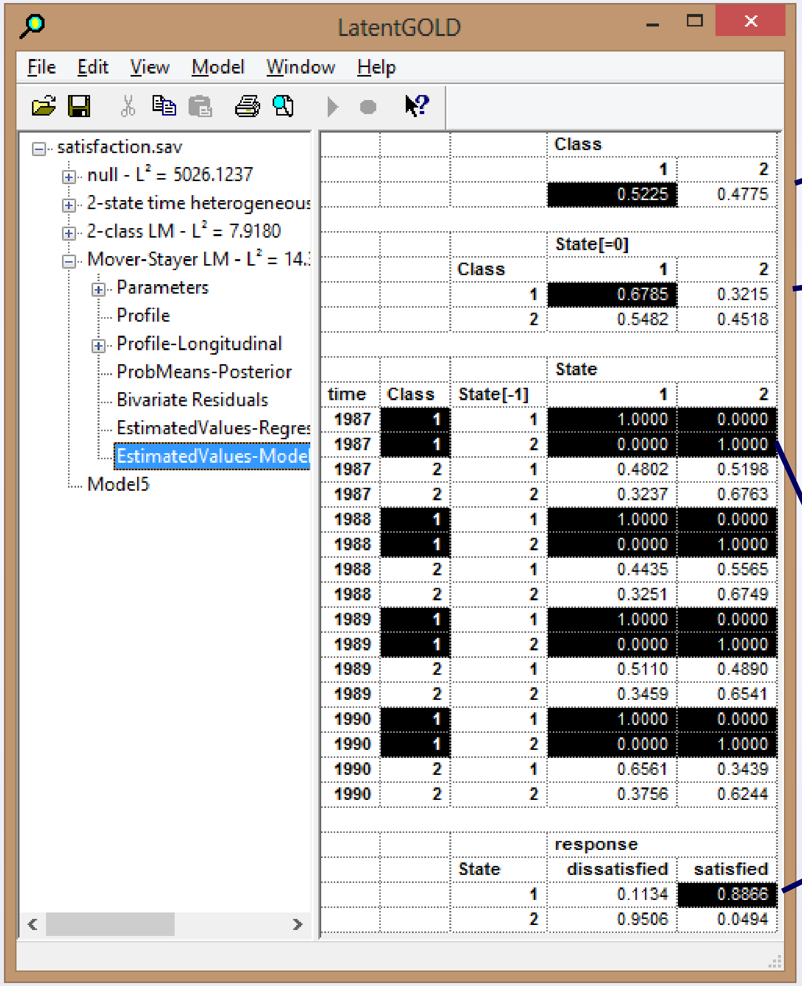
Initial State

Transition Probabilities

Measurement model

Example 2: Mover-Stayer Time-heterogeneous Latent Markov Model

The Estimated Values output shows that 52.25% of respondents are in the Stayer class, who tend to be mostly Satisfied with their lives throughout this 5 year period -- 67.85% are in state 1 ('Satisfied' state) initially and remain in that state. In contrast, among respondents whose life satisfaction *changed* during this 5 year period (the 'Mover' class), fewer (54.82%) were in the Satisfied state during the initial year.



	Class 1	Class 2
	0.5225	0.4775

Class	State[=0] 1	State[=0] 2
1	0.6785	0.3215
2	0.5482	0.4518

time	Class	State[-1]	State 1	State 2
1987	1	1	1.0000	0.0000
1987	1	2	0.0000	1.0000
1987	2	1	0.4802	0.5198
1987	2	2	0.3237	0.6763
1988	1	1	1.0000	0.0000
1988	1	2	0.0000	1.0000
1988	2	1	0.4435	0.5565
1988	2	2	0.3251	0.6749
1989	1	1	1.0000	0.0000
1989	1	2	0.0000	1.0000
1989	2	1	0.5110	0.4890
1989	2	2	0.3459	0.6541
1990	1	1	1.0000	0.0000
1990	1	2	0.0000	1.0000
1990	2	1	0.6561	0.3439
1990	2	2	0.3756	0.6244

State	response dissatisfied	response satisfied
1	0.1134	0.8866
2	0.9506	0.0494

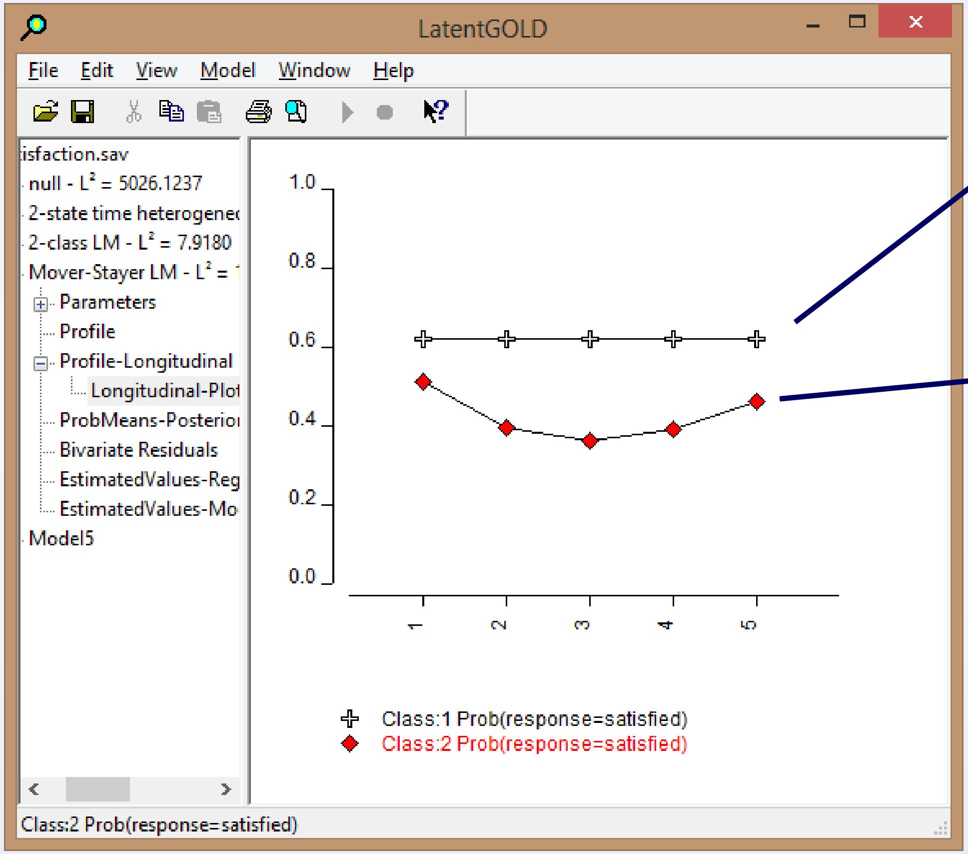
Class Size

Initial State

Transition Probabilities – note that class 1 probability of staying in the same state has been restricted to 1.

Measurement model

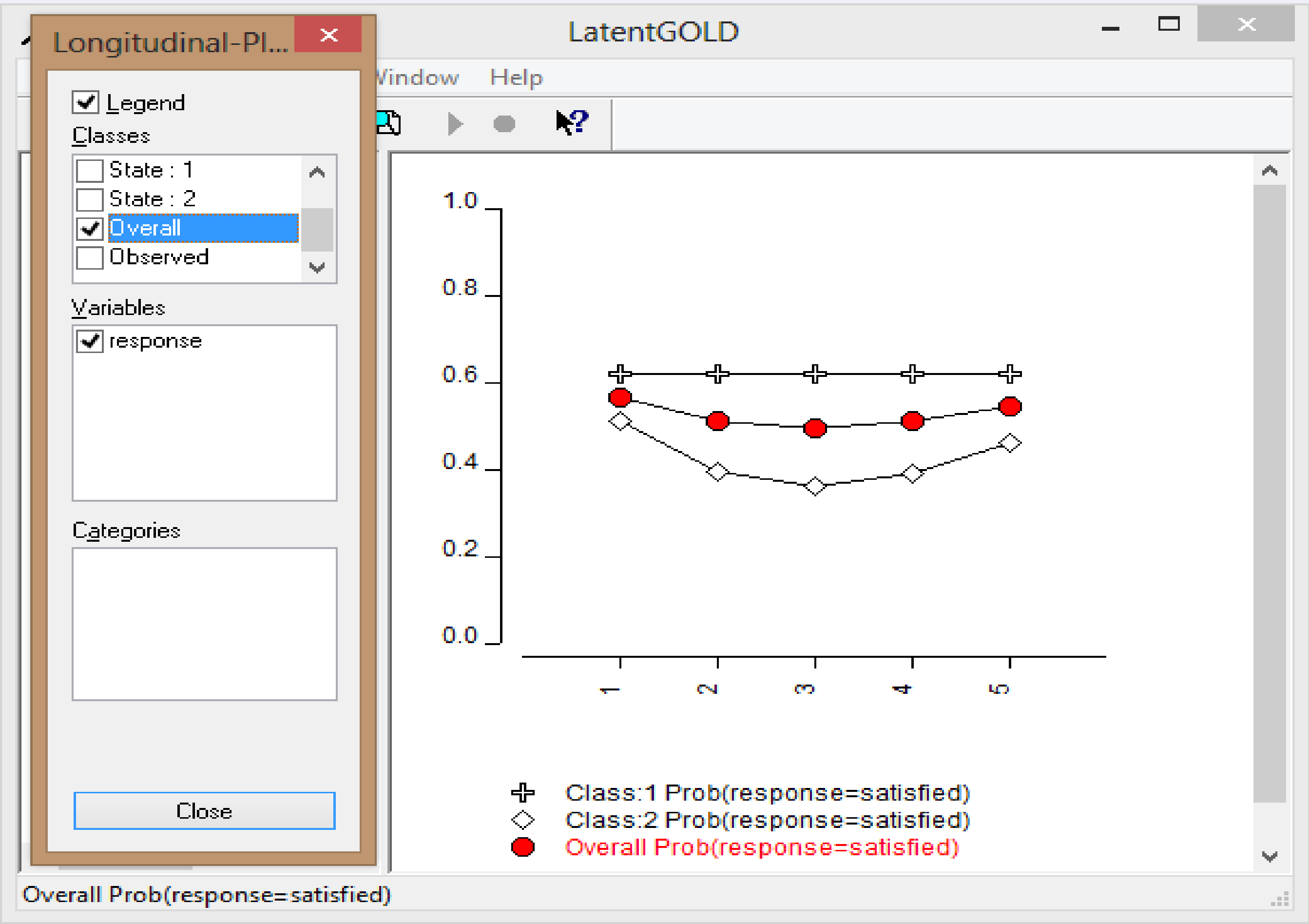
Example 2: Longitudinal Profile Plot for the Mover-Stayer LM model



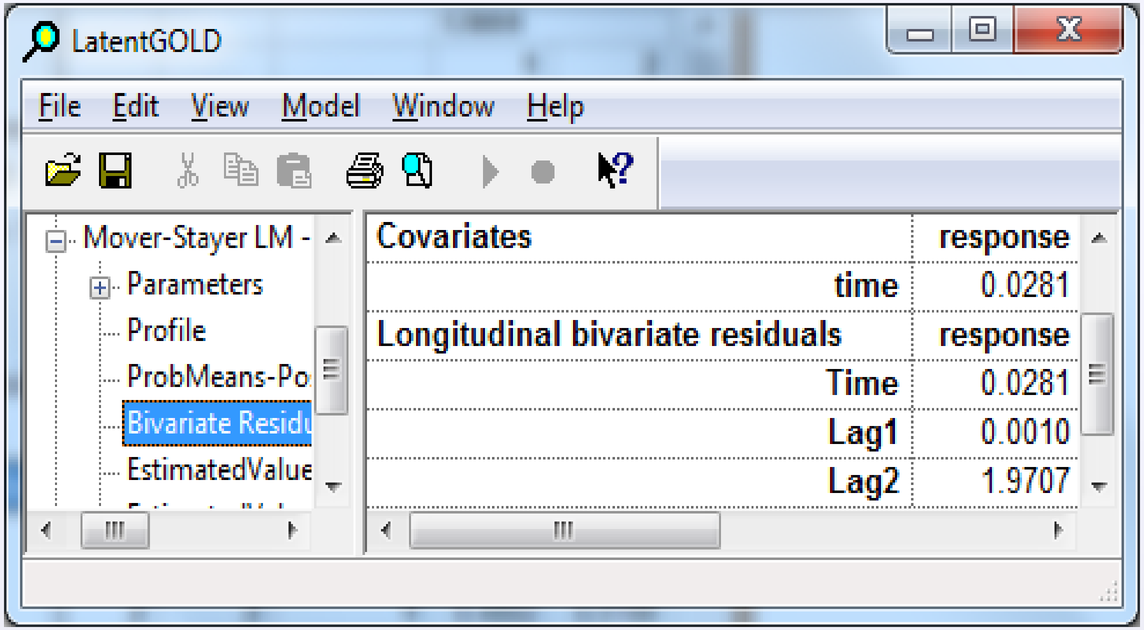
Stayer class showing 61.75% satisfied each year

Mover class showing changes over time

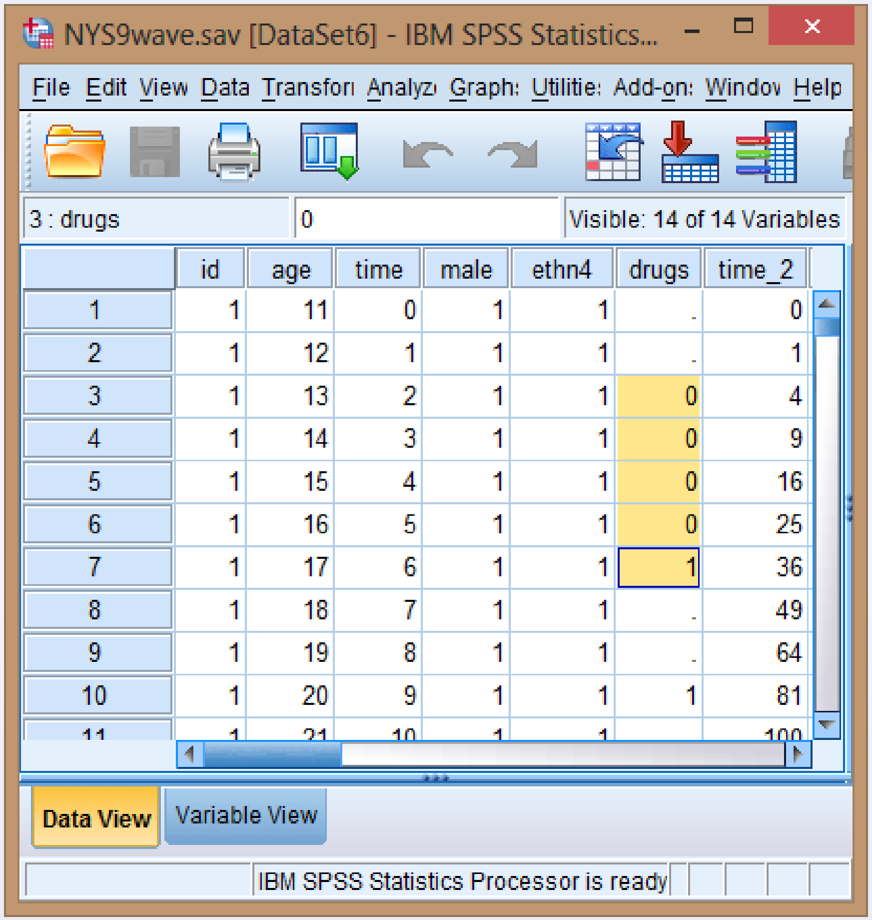
Example 2: Longitudinal-Plot with Predicted Probability of Being Satisfied ('Overall Prob') Appended



Example 2: L-BVRs for the Mover-Stayer Model



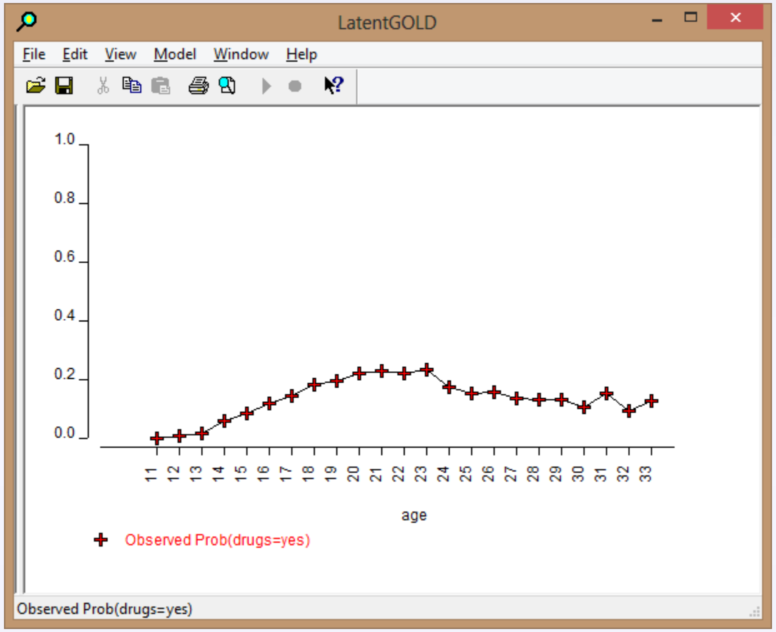
Example 3: Latent GOLD Longitudinal Analysis of Sparse Data

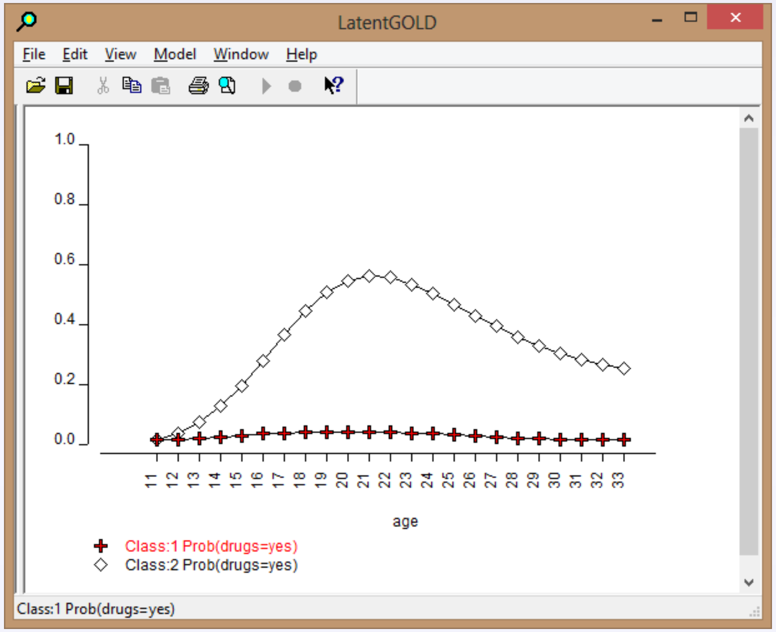


	id	age	time	male	ethn4	drugs	time_2
1	1	11	0	1	1	.	0
2	1	12	1	1	1	.	1
3	1	13	2	1	1	0	4
4	1	14	3	1	1	0	9
5	1	15	4	1	1	0	16
6	1	16	5	1	1	0	25
7	1	17	6	1	1	1	36
8	1	18	7	1	1	.	49
9	1	19	8	1	1	.	64
10	1	20	9	1	1	1	81
11	1	21	10	1	1		100

- N=1725 pupils who were of age 11-17 at the initial measurement occasion (in 1976)
- Survey conducted annually from 1976 to 1980 and at three year intervals after 1980
- 23 time points (T+1=23), where t=0 corresponds to age 11 and the last time point to age 33.
- For each subject, data is observed for at most 9 time points (the average is 7.93) which means that responses for the other time points are treated as missing. (See Figure 2)
- Dichotomous dependent variable – ‘drugs’ indicating whether respondent used hard drugs during the past year (1=yes; 0=no).
- Time-varying predictors are ‘time’ (t) and ‘time_2’ (t^2); time-constant predictors are ‘male’ and ‘ethn4’ (ethnicity).

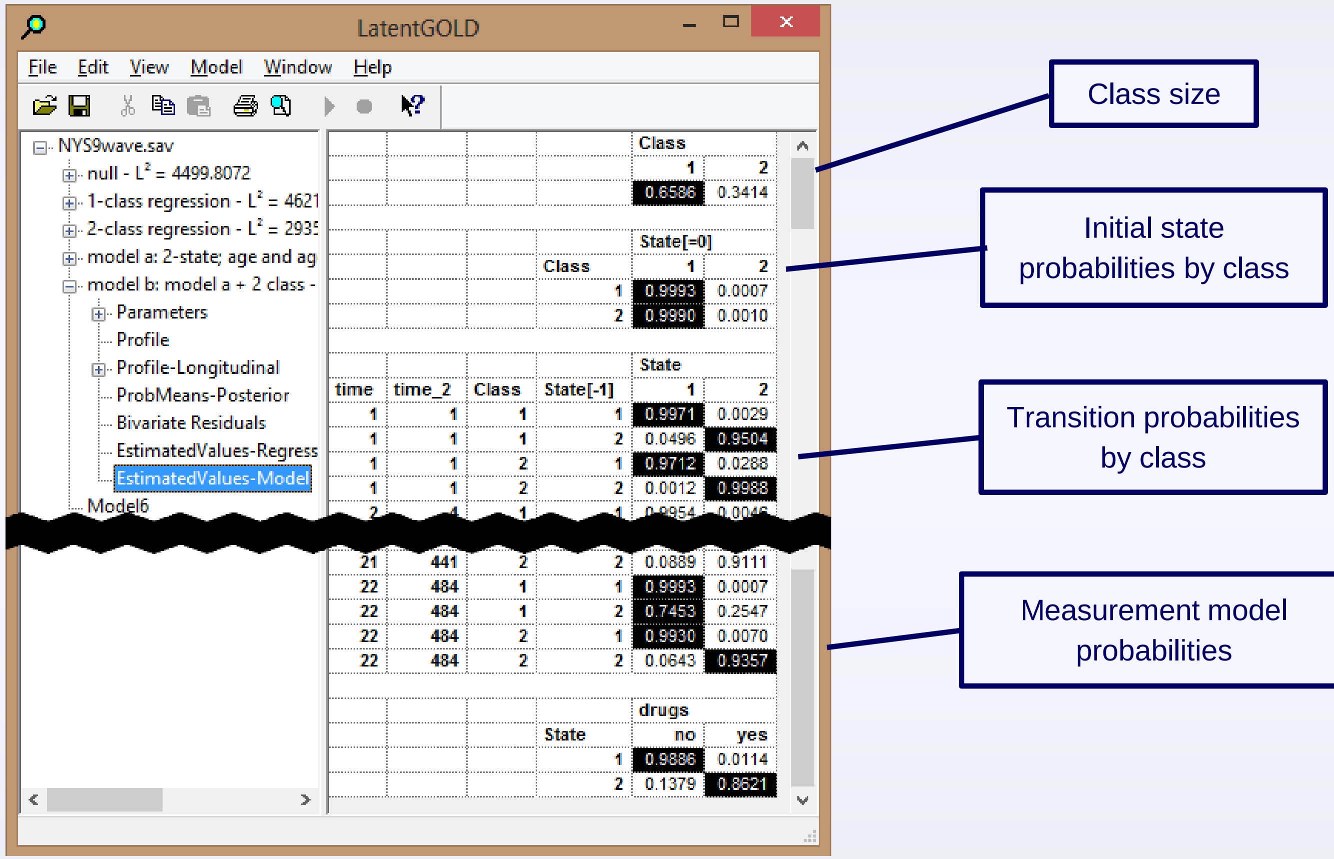
Example 3: Latent GOLD Longitudinal Analysis of Sparse Data



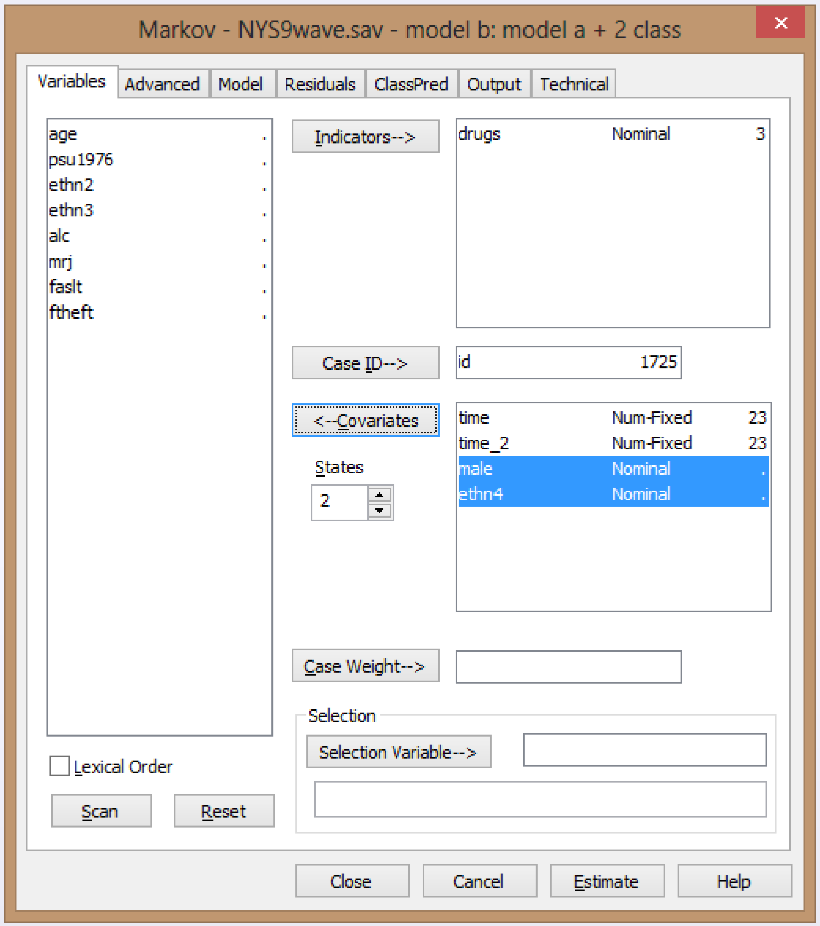


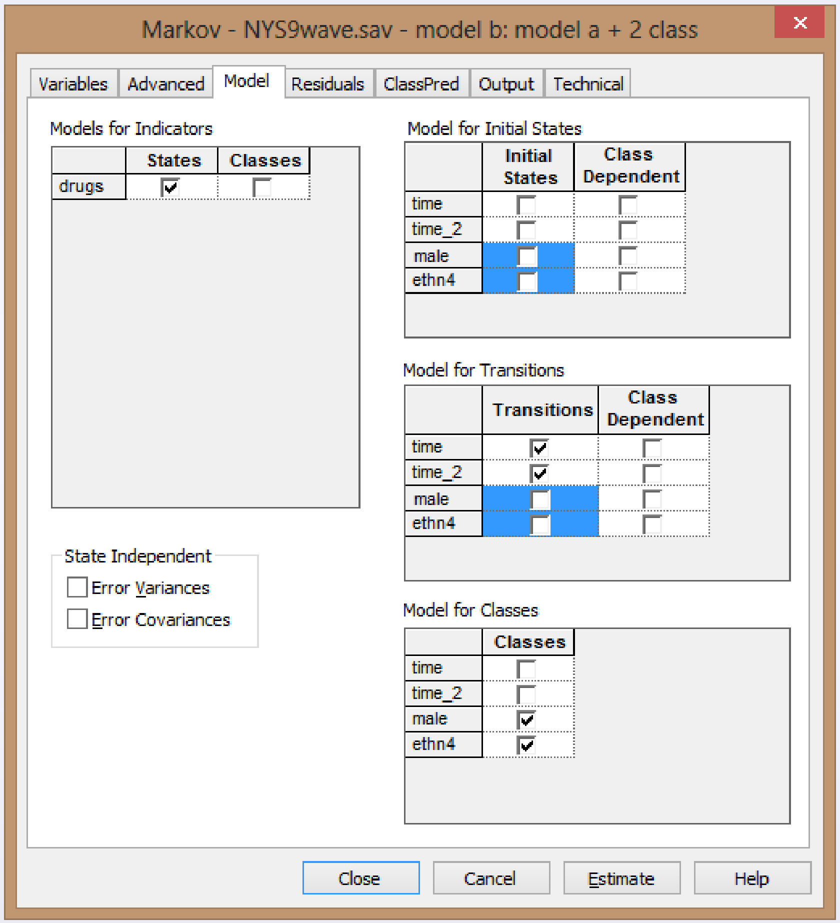
The plot on the left shows the overall trend in drug usage during this period is non-linear, with zero usage reported for 11 year olds, increasing to a peak in the early 20s and then declining through age 33. The plot on the right plots the results from a mixture latent Markov model suggesting that the population consists of 2 distinct segments with different growth rates, Class 2 consisting primarily of non-users.

Example 3: 2-class MLM Model

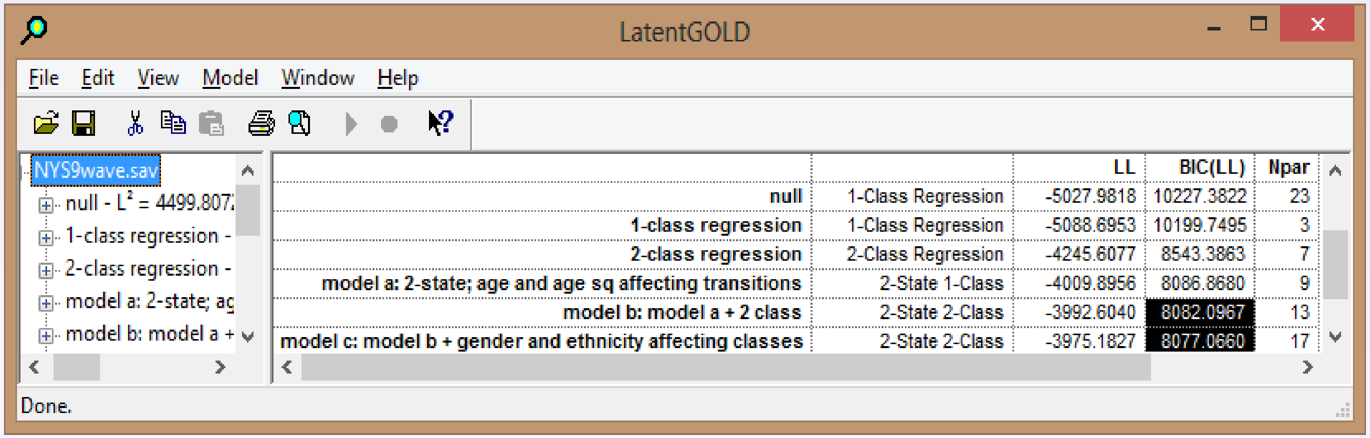


Example 3: Including Gender and Ethnicity as Covariates in Model





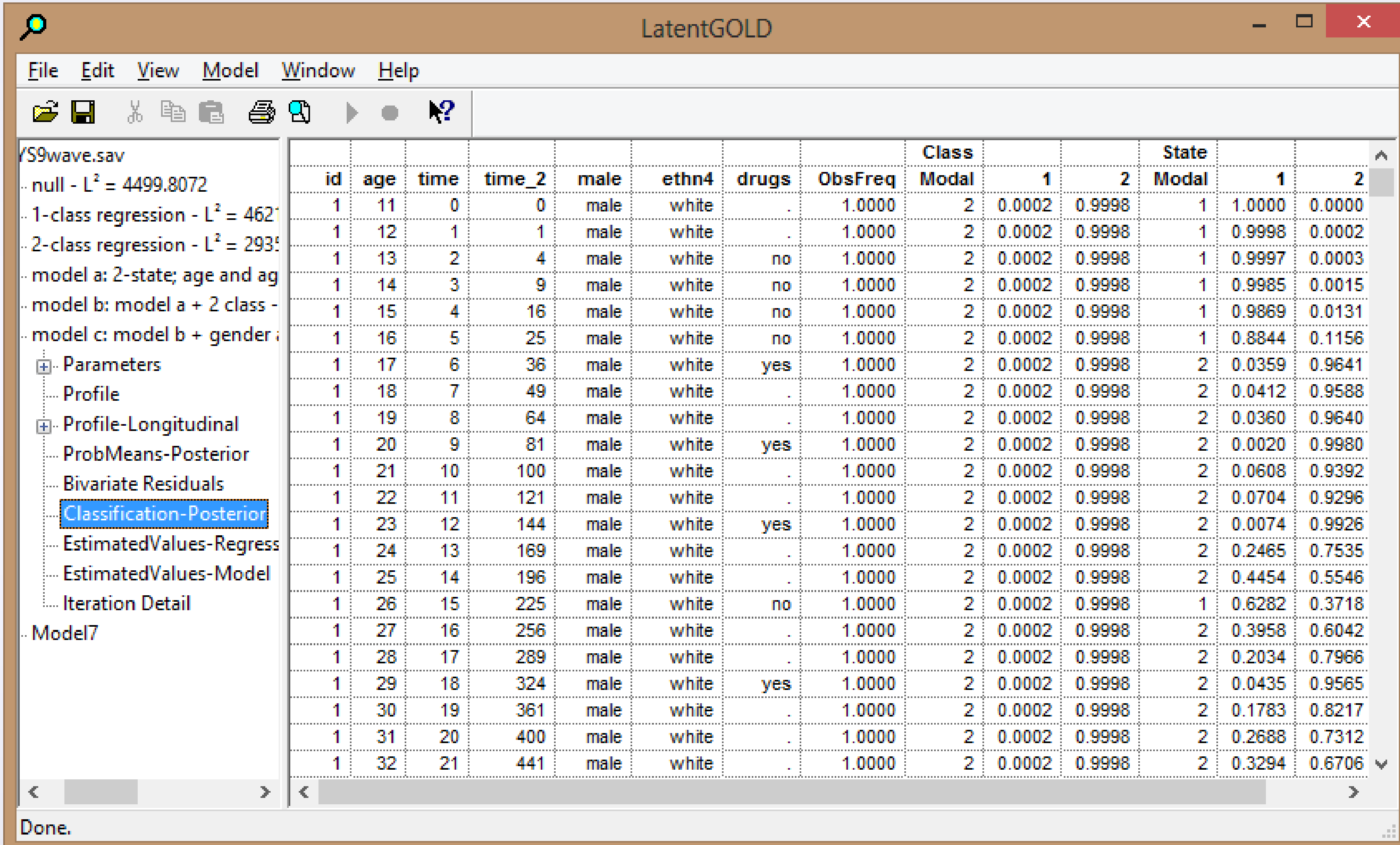
Example 3: Including Gender and Ethnicity as Covariates in Model



		LL	BIC(LL)	Npar
null	1-Class Regression	-5027.9818	10227.3822	23
1-class regression	1-Class Regression	-5088.6953	10199.7495	3
2-class regression	2-Class Regression	-4245.6077	8543.3863	7
model a: 2-state; age and age sq affecting transitions	2-State 1-Class	-4009.8956	8086.8680	9
model b: model a + 2 class	2-State 2-Class	-3992.6040	8082.0967	13
model c: model b + gender and ethnicity affecting classes	2-State 2-Class	-3975.1827	8077.0660	17

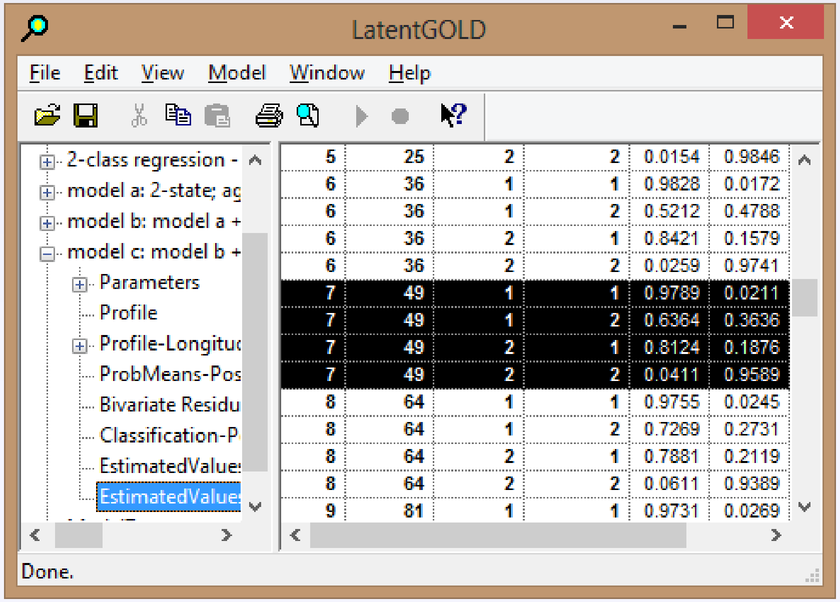
Model Summary output, showing that adding gender and ethnicity improves the fit.

Example 3: Including Gender and Ethnicity as Covariates in Model



								Class			State		
id	age	time	time_2	male	ethn4	drugs	ObsFreq	Modal	1	2	Modal	1	2
1	11	0	0	male	white	.	1.0000	2	0.0002	0.9998	1	1.0000	0.0000
1	12	1	1	male	white	.	1.0000	2	0.0002	0.9998	1	0.9998	0.0002
1	13	2	4	male	white	no	1.0000	2	0.0002	0.9998	1	0.9997	0.0003
1	14	3	9	male	white	no	1.0000	2	0.0002	0.9998	1	0.9985	0.0015
1	15	4	16	male	white	no	1.0000	2	0.0002	0.9998	1	0.9869	0.0131
1	16	5	25	male	white	no	1.0000	2	0.0002	0.9998	1	0.8844	0.1156
1	17	6	36	male	white	yes	1.0000	2	0.0002	0.9998	2	0.0359	0.9641
1	18	7	49	male	white	.	1.0000	2	0.0002	0.9998	2	0.0412	0.9588
1	19	8	64	male	white	.	1.0000	2	0.0002	0.9998	2	0.0360	0.9640
1	20	9	81	male	white	yes	1.0000	2	0.0002	0.9998	2	0.0020	0.9980
1	21	10	100	male	white	.	1.0000	2	0.0002	0.9998	2	0.0608	0.9392
1	22	11	121	male	white	.	1.0000	2	0.0002	0.9998	2	0.0704	0.9296
1	23	12	144	male	white	yes	1.0000	2	0.0002	0.9998	2	0.0074	0.9926
1	24	13	169	male	white	.	1.0000	2	0.0002	0.9998	2	0.2465	0.7535
1	25	14	196	male	white	.	1.0000	2	0.0002	0.9998	2	0.4454	0.5546
1	26	15	225	male	white	no	1.0000	2	0.0002	0.9998	1	0.6282	0.3718
1	27	16	256	male	white	.	1.0000	2	0.0002	0.9998	2	0.3958	0.6042
1	28	17	289	male	white	.	1.0000	2	0.0002	0.9998	2	0.2034	0.7966
1	29	18	324	male	white	yes	1.0000	2	0.0002	0.9998	2	0.0435	0.9565
1	30	19	361	male	white	.	1.0000	2	0.0002	0.9998	2	0.1783	0.8217
1	31	20	400	male	white	.	1.0000	2	0.0002	0.9998	2	0.2688	0.7312
1	32	21	441	male	white	.	1.0000	2	0.0002	0.9998	2	0.3294	0.6706

Example 3: Including Gender and Ethnicity as Covariates in Model



For concreteness we will focus on 18 year olds (see highlighted cells in figure). We see that 18 year olds who were in the lower usage state (State 1) at age 17 have a probability of .1876 of switching to the higher usage state (State 2) if they are in Class 2 compared to a probability of only .0211 of switching if they were in Class 1. In addition, if they were in the higher use state (State 2) at age 17, they have a probability of .9589 of remaining in that state compared to only .3636 if they were in Class 1. Thus, based on these different transition probabilities we see that Class 2 is more likely to move to and remain in a higher drug usage state than Class 1.

Appendix: Equations

Latent Markov:

$$P(\mathbf{y}_i) = \sum_{x_0=1}^{K}\sum_{x_1=1}^{K}\dots\sum_{x_T=1}^{K} P(x_0, x_1, \dots, x_T) P(\mathbf{y}_i \mid x_0, x_1, \dots, x_T)$$

$$P(x_0, x_1, \dots, x_T) = P(x_0)\prod_{t=1}^{T} P(x_t \mid x_{t-1})$$

$$P(\mathbf{y}_i \mid x_0, x_1, \dots, x_T) = \prod_{t=0}^{T} P(\mathbf{y}_{it} \mid x_t) = \prod_{t=0}^{T}\prod_{j=1}^{J} P(y_{itj} \mid x_t)$$

Latent Growth:

$$P(\mathbf{y}_i \mid \mathbf{z}_i) = \sum_{w=1}^{L} P(w \mid \mathbf{z}_i) \prod_{t=0}^{T} P(y_{it} \mid w, \mathbf{z}_{it})$$

References

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