

# Latent GOLD 4.0 and IRT modeling

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Latent GOLD 4.0 can be used to obtain MML (Marginal Maximum Likelihood) estimates for a large variety of well-known IRT models, such as the Rasch or 1PL model and the Birnbaum or 2PL model for dichotomous items, the Partial Credit model, the Rating Scale model for ordinal items, and the Nominal Response model for nominal items. Standard parametric IRT models can be defined using a CFactor (continuous factor or latent trait) in a Cluster or Regression model with a single latent class. Note that the CFactor option is implemented only in Latent GOLD 4.0 Advanced. By increasing the number of latent classes one obtains mixture variants of IRT models, such as Rost's mixed Rasch model. Rather than assuming that the trait is continuous and normally distributed, one can assume that it is discrete as was proposed among others by Heinen. This is achieved with the so-called discrete factor (DFactor) module.

Additional features are that *hybrid* models can be defined for items of mixed scale types, including counts and continuous items. Moreover, multi-dimensional models can be estimated, and the local independence assumption can be tested and relaxed by including direct effects between items. The Regression module makes it possible to define a general class of linear logistic test models.

While we do not discuss *all* extended features, in this article we focus on how the more standard unidimensional IRT models are implemented in Latent GOLD. In particular, we show how to transform Latent GOLD's logistic parameters to obtain standard IRT parameters.

## 1 Rasch model for dichotomous items

The standard formulation of the Rasch model is

$$\log \frac{P(Y_t = 1|\theta_i)}{P(Y_t = 0|\theta_i)} = \theta_i - \delta_t,$$

where  $i$  is a person index,  $t$  an item index,  $\theta_i$  a person's trait value, and  $\delta_t$  an item difficulty. In MML estimation, besides the  $\delta_t$  we estimate the variance and mean of  $\theta_i$ . For identification, we must either set  $E(\theta_i) = 0$ , fix one  $\delta_t$  to 0, or let the  $\delta_t$ s sum to 0.

The binary logistic item response model implemented in Latent GOLD Cluster for one CFactor with "Equal Effects" and dichotomous items defined to be "ordinal" is:

$$\log \frac{P(Y_t = 1|F_i)}{P(Y_t = 0|F_i)} = (\beta_{01}^t - \beta_{00}^t) + \lambda F_i,$$

with  $E(F_i) = 0$  and  $Var(F_i) = 1$ . The link between this equation and the above Rasch model is rather simple:

$$\begin{aligned} \delta_t &= -(\beta_{01}^t - \beta_{00}^t) \\ E(\theta_i) &= 0 \\ Var(\theta_i) &= \lambda^2 \end{aligned}$$

The relationship is even simpler if we use the Output Tab to change from the effects coding default to dummy coding. With the first category as reference category,  $\beta_{00}^t = 0$  and  $\delta_t = -\beta_{01}^t$ , and with the last category as reference category,  $\beta_{01}^t = 0$  and  $\delta_t = \beta_{00}^t$ .

In the Latent GOLD Regression module it is possible to use "item number" as a nominal predictor (note that the data should be in long-file or two-level format). With effects coding, the corresponding item parameters will sum to 0, which means that  $E(\theta_i)$  can be estimated. The regression model with one CFactor is

$$\log \frac{P(Y_t = 1|F_i)}{P(Y_t = 0|F_i)} = (\beta_{01} - \beta_{00}) + \beta_{1t} + \lambda F_i,$$

The link with the standard Rasch formulation is:

$$\begin{aligned} \delta_t &= -\beta_{1t} \\ E(\theta_i) &= (\beta_{01} - \beta_{00}) \\ Var(\theta_i) &= \lambda^2 \end{aligned}$$

The Latent GOLD Regression parameters (and, for example the parameters output by the WinMira program) can be transformed to those obtained from

the Latent GOLD Cluster module by subtracting the average  $\delta_t$  ( $\bar{\delta}$ ) from each  $\delta_t$  and from  $E(\theta_i)$ . To reverse the process, the Cluster parameters can be obtained from the corresponding Regression (and WinMira) parameters by subtracting  $E(\theta_i)$  from each  $\delta_t$  and from  $E(\theta_i)$ . It is a matter of personal preference whether to assume that the thresholds  $\delta_t$  are 0 on average or that  $E(\theta_i)$  equals 0.

## 2 Mixture Rasch model for dichotomous items

The mixture Rasch model estimated with the Latent GOLD Cluster procedure has the following form

$$\log \frac{P(Y_t = 1|F_i, x)}{P(Y_t = 0|F_i, x)} = (\beta_{01}^t - \beta_{00}^t) + \beta_{1x}^t + \lambda_x F_i,$$

assuming that the CFactor effects are equal across items, but unequal across classes. For identification we set the  $\beta_{1x}^t$  for one class  $x$  to 0 (dummy coding) or let the  $\beta_{1x}^t$  sum to 0 across classes (effect coding). For class  $x$ ,

$$\begin{aligned} \delta_{tx} &= -(\beta_{01}^t - \beta_{00}^t) - \beta_{1x}^t \\ E(\theta_i|x) &= 0 \\ Var(\theta_i|x) &= \lambda_x^2 \end{aligned}$$

In the Latent GOLD Regression parameterization with all effects class dependent, we get

$$\log \frac{P(Y_t = 1|F_i, x)}{P(Y_t = 0|F_i, x)} = (\beta_{01x} - \beta_{00x}) + \beta_{1tx} + \lambda_x F_i,$$

which implies that

$$\begin{aligned} \delta_{tx} &= -\beta_{1tx} \\ E(\theta_i|x) &= (\beta_{01x} - \beta_{00x}) \\ Var(\theta_i|x) &= \lambda_x^2 \end{aligned}$$

## 3 Partial Credit model for ordinal items

The standard formulation of the PCM is

$$\log \frac{P(Y_t = m|\theta_i)}{P(Y_t = m-1|\theta_i)} = \theta_i - \delta_{tm},$$

where  $m$  goes from 0 to the number of categories minus one. In LG Cluster, a CFactor model with equal effects across items yields a logistic regression model for ordinal items of the form

$$\log \frac{P(Y_t = m | F_i)}{P(Y_t = m - 1 | F_i)} = (\beta_{0m}^t - \beta_{0m-1}^t) + \lambda \cdot F_i.$$

The connection between the two parametrizations is

$$\begin{aligned} \delta_{tm} &= -(\beta_{0m}^t - \beta_{0m-1}^t) \\ E(\theta_i) &= 0 \\ Var(\theta_i) &= \lambda^2 \end{aligned}$$

The program WinMira assumes  $\sum_{t=1}^T \sum_{m=2}^M \delta_{tm} = 0$  and estimates  $E(\theta_i)$  freely. If we subtract the average  $\delta_{tm}$  across items and item steps ( $\bar{\delta}$ ) from each  $\delta_{tm}$ , we get the WinMira  $\delta_{tm}^{WM}$ . The term  $E^{WM}(\theta_i)$  becomes equal to the negative of this number. In formulas:

$$\begin{aligned} \delta_{tm}^{WM} &= \delta_{tm} - \bar{\delta} \\ E^{WM}(\theta_i) &= -\bar{\delta} \\ Var^{WN}(\theta_i) &= Var^{WN}(\theta_i) \end{aligned}$$

It should be noted that WinMira also reports  $\sum_{m=2}^M \delta_{tm}^{WM} / (M - 1)$  and calls these item locations (probably for comparability with the rating scale model, see below). Again, the only issue involved is whether one prefers that thresholds sum to 0 or that  $E(\theta_i)$  equals 0.

In its mixture form (when the number of clusters is 2 or larger), we get

$$\log \frac{P(Y_t = m | F_i, x)}{P(Y_t = m - 1 | F_i, x)} = (\beta_{0m}^t - \beta_{0m-1}^t) + \beta_{1x}^t + \lambda_x \cdot F_i,$$

where now

$$\begin{aligned} \delta_{tmx} &= -(\beta_{0m}^t - \beta_{0m-1}^t) - \beta_{1x}^t \\ E(\theta_i | x) &= 0 \\ Var(\theta_i | x) &= \lambda_x^2 \end{aligned}$$

This shows that Latent GOLD 4.0 does not implement a fully unrestricted mixture PCM, but a model in which the classes differ which respect to a shift in item difficulty that is common to all categories of the item concerned (a shift in location).

## 4 Rating Scale model for ordinal items

The standard formulation of the RSM is

$$\log \frac{P(Y_t = m|\theta_i)}{P(Y_t = m - 1|\theta_i)} = \theta_i - \delta_t - \tau_m,$$

In LG Regression, a model with “item number” as the nominal predictor and with a CFactor yields an ordinal logistic regression model of the form

$$\log \frac{P(Y_t = m|F_i)}{P(Y_t = m - 1|F_i)} = (\beta_{0m} - \beta_{0m-1}) + \beta_{1t} + \lambda \cdot F_i.$$

The connection between the two parameterizations is

$$\begin{aligned} \tau_m &= -(\beta_{0m} - \beta_{0m-1}) \\ \delta_t &= -\beta_{1t} \\ E(\theta_i) &= 0 \\ Var(\theta_i) &= \lambda^2 \end{aligned}$$

In WinMira,  $\sum_t^T \delta_t = 0$ ,  $\sum_m^M \tau_m = 0$ , and  $E(\theta_i)$  is a free parameter. The constraint  $\sum_t^T \delta_t = 0$  is also imposed by LG when effect coding is used, but not under dummy coding. To get the WinMira  $\tau_m$  you have to subtract the average  $\tau_m$  ( $-\bar{\tau}$ ) from each  $\tau_m$ , and  $E(\theta_i)$  is the negative of this average. In general terms (also for dummy coding thus),

$$\begin{aligned} \tau_m^{WM} &= \tau_m - \bar{\tau} \\ \delta_t^{WM} &= \delta_t - \bar{\delta} \\ E(\theta_i) &= -\bar{\delta} - \bar{\tau} \\ Var^{WM}(\theta_i) &= Var(\theta_i) \end{aligned}$$

Note that WinMira reports  $\delta_t^{WM}$  (item locations) and  $\delta_t^{WM} + \tau_m^{WM}$  (threshold parameters). Instead of the separate  $\tau_m^{WM}$ , WinMira reports  $\tau_m^{WM} - \tau_{m-1}^{WM}$  as mean threshold distances.

In its mixture form, we get

$$\log \frac{P(Y_t = m|F_i, x)}{P(Y_t = m - 1|F_i, x)} = (\beta_{0mx} - \beta_{0m-1x}) + \beta_{1tx} + \lambda_x \cdot F_i,$$

where now

$$\begin{aligned}\tau_{mx} &= -(\beta_{0mx} - \beta_{0m-1x}) \\ \delta_{tx} &= -\beta_{1tx} \\ E(\theta_i|x) &= 0 \\ Var(\theta_i|x) &= \lambda_x^2\end{aligned}$$

This shows that, contrary to what we saw for the PCM, LG 4.0 Regression implements a fully unrestricted mixture RSM.

## 5 Discretized Rasch-type models

Rather than assuming that the trait is a continuous normally distributed variable, one can also work with a discretized trait with  $K$  levels, classes, or nodes. In DFactor, these  $K$  nodes are assumed to have fixed equidistant locations: the lowest class has location 0 ( $x_1 = 0$ ), and the highest class has location 1 ( $x_K = 1$ ). The weights  $\pi_k$  (the class sizes) are freely estimated.

A discretized Rasch or Partial credit model is obtained by indicating that the DFactor has “Equal Effects” on the items (the number of levels of the DFactor can be varied by the user). The resulting DFactor model is

$$\log \frac{P(Y_t = m|x_k)}{P(Y_t = m-1|x_k)} = (\beta_{0m}^t - \beta_{0m-1}^t) + \beta \cdot x_k.$$

If we translated this into the Rasch or PCM parameters, we get

$$\begin{aligned}\delta_{tm} &= -(\beta_{0m}^t - \beta_{0m-1}^t) \\ E(\theta_i) &= \beta \cdot \sum_{k=1}^K x_k \cdot \pi_k \\ Var(\theta_i) &= \sum_{k=1}^K \beta^2 \cdot x_k^2 \cdot \pi_k - [E(\theta_i)]^2.\end{aligned}$$

Note that the same types of transformation on the  $\delta_{tm}$  parameters as discussed above can be applied if one prefers a parameterization in which  $E(\theta_i) = 0$  and  $Var(\theta_i) = 1$ .

## 6 Other types of IRT models

Three other types of IRT models that can easily be defined with the Latent GOLD Cluster procedure using one CFactor are the

- Birnbaum or 2-PL model for dichotomous items,
- Generalized Partial Credit model for ordinal items,
- Nominal Response model.

These models differ from the Rasch model in that they contain a discrimination parameter  $\alpha$  for each item or each item category. In the IRT literature, the latter two models are usually defined as follows:

$$\begin{aligned}\log \frac{P(Y_t = m|\theta_i)}{P(Y_t = m-1|\theta_i)} &= \alpha_t (\theta_i - \delta_{tm}) \\ \log \frac{P(Y_t = m|\theta_i)}{P(Y_t = 0|\theta_i)} &= \alpha_{tm} (\theta_i - \delta_{tm}),\end{aligned}$$

where the category index  $m$  goes again from 0 to the number of categories minus one. Note that the Birnbaum model is not defined separately because it is a special case for dichotomous items in either of these two equations. For identification, one usually sets  $E(\theta_i) = 0$  and  $Var(\theta_i) = 1$ .

In Latent GOLD Cluster, the same two models are defined as follows:

$$\begin{aligned}\log \frac{P(Y_t = m|\theta_i)}{P(Y_t = m-1|\theta_i)} &= (\beta_{0m}^t - \beta_{0m-1}^t) + \lambda^t F_i, \\ \log \frac{P(Y_t = m|\theta_i)}{P(Y_t = 0|\theta_i)} &= (\beta_{0m}^t - \beta_{00}^t) + \lambda_m^t F_i.\end{aligned}$$

It is easy to verify that

$$\begin{aligned}\alpha_t &= \lambda^t, \\ \delta_{tm} &= -(\beta_{0m}^t - \beta_{0m-1}^t)/\lambda^t,\end{aligned}$$

for the Generalized Partial Credit model, and that

$$\begin{aligned}\alpha_{tm} &= (\lambda_m^t - \lambda_0^t), \\ \delta_{tm} &= -(\beta_{0m}^t - \beta_{00}^t)/(\lambda_m^t - \lambda_0^t),\end{aligned}$$

for the Nominal Response model. Note that the transformation formulas for the Nominal Response model can be simplified by using dummy coding with the first category,  $Y_t = 0$ , as the reference category. In that case, the terms  $\beta_{00}^t$  and  $\lambda_0^t$  are restricted to be equal to 0 and cancel from the above equations.

Unrestricted mixture variants can be defined by increasing the number of clusters and indicating that the ‘‘CFactor Effects’’ are not class independent. The models change in ways similar to what was discussed for the Rasch-type models. The discrimination parameters will be class specific and there will be an extra term  $\beta_{1x}^t$  ( $\beta_{1mx}^t$ ) in the formula for the class-specific item-category difficulties.

Discretized variants of these models using the DFactor module have the following form:

$$\begin{aligned} \log \frac{P(Y_t = m|x_k)}{P(Y_t = m - 1|x_k)} &= (\beta_{0m}^t - \beta_{0m-1}^t) + \beta^t \cdot x_k, \\ \log \frac{P(Y_t = m|x_k)}{P(Y_t = 0|x_k)} &= (\beta_{0m}^t - \beta_{00}^t) + \beta_m^t \cdot x_k. \end{aligned}$$

In both cases,  $E(\theta_i) = \sum_{k=1}^K x_k \cdot \pi_k$  and  $Var(\theta_i) = \sum_{k=1}^K x_k^2 \cdot \pi_k - [E(\theta_i)]^2$ . In the computation of the item-category difficulties,  $\beta^t$  ( $\beta_m^t$ ) takes the role of  $\lambda^t$  ( $\lambda_m^t$ ) in the above transformation formulas.