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On the statistical and theoretical basis of signal detection theory and extensions: Unequal variance, random coefficient, and mixture models

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ABSTRACT

Basic results for conditional means and variances, as well as distributional results, are used to clarify the similarities and differences between various extensions of signal detection theory (SDT). It is shown that a previously presented motivation for the unequal variance SDT model (varying strength) actually leads to a related, yet distinct, model. The distinction has implications for other extensions of SDT, such as models with criteria that vary over trials. It is shown that a mixture extension of SDT is also consistent with unequal variances, but provides a different interpretation of the results; mixture SDT also offers a way to unify results found across several types of studies.

Signal detection theory (SDT; Green & Swets, 1988; Macmillan & Creelman, 2005; Wickens, 2002) views detection or discrimination tasks as consisting of two basic components — a perceptual component, which has to do with the effect of presenting a signal on an observer's perception, and a decision component, which has to do with an observer's use of response criteria. A useful aspect of SDT is that it separates these two components, whereas they were often confounded in earlier research. Associated with the perceptual component is a measure d, which has an interpretation in terms of the distance between underlying perceptual distributions for signal and noise, which reflects an observer's ability to discriminate. Associated with the decision component is a measure c_k, which has an interpretation in terms of the locations of $K-1$ response criteria, which reflect the ways in which an observer tends to use the different response categories.

The basic SDT model, with equal variances for signal and noise, predicts that receiver operating characteristic (ROC) curves on inverse normal coordinates, that is, z-ROC curves, will have slopes of unity (for the normal SDT model), because the z-ROC slope is equal to the ratio of noise to signal standard deviations, σ_n/σ_s. However, z-ROC curves with slopes other than unity have often been found (see Macmillan & Creelman, 2005; Swets, 1986). To handle z-ROC curves with slopes other than unity, Green and Swets (1988) introduced an unequal variance extension of SDT. The unequal variance extension introduces a parameter that allows the variances of the underlying distributions to differ across signal and noise. As noted by Green and Swets, the unequal variance SDT model was not theoretically motivated, but was an empirical generalization, in that a parameter was introduced to simply improve fit. As a result, although the unequal variance SDT model has been widely used, questions remain as to the interpretation of the variance parameter.

The present article examines two extensions of SDT that each provide a basis for the unequal variance model. The first is a model where d varies over trials, which will be referred to simply as the *varying strength* model, in line with earlier ideas. It is shown that, although the varying strength model has been suggested as being equivalent to the unequal variance SDT model, it is formally distinct from it. The distinction is shown to make a difference when models with criteria that randomly shift over trials are considered. It is shown that, contrary to some claims, varying criteria do not necessarily change the SDT model in any fundamental way, depending on how the model is formalized. The second approach to unequal variances is provided by a mixture extension of SDT. The mixture approach offers a different interpretation of the results; it also provides a way to unify results found across different procedures, such as recognition and discrimination, in that it can account for different types of non-normality.

It is shown that all of the models are concisely summarized in terms of their implied conditional means, variances, and

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distributions. The results are illustrated with examples from memory research, where SDT has been extensively applied; however, the results are general and can be applied to any area where SDT has been applied. The derivations provide a statistical and theoretical foundation that clarifies the consequences of basic extensions of SDT.

1. The equal variance SDT model

1.1. The statistical model

From a statistical perspective, the equal variance SDT model is a model of cumulative response probabilities conditional on observed variables; the use of cumulative probabilities is one way to take the ordinal nature of the responses into account (see Agresti, 2002). More specifically, let the response variable be denoted by Y, which takes on discrete values k with $1 \leq k \leq K$, and let X be a dichotomous variable with values x of 0 (for noise) or 1 (for signal). For normal underlying distributions, the equal variance SDT model can be written as

$$p(Y \leq k \mid x) = \Phi(c_k - dx), \qquad (1)$$

where the first term is the cumulative probability of a response of k or less for Y conditional on X, d is the distance between the signal and noise distributions, c_k are $K - 1$ response criteria for the K response categories, and Φ is the cumulative distribution function (CDF) for the normal distribution. Note that the model can be formulated with distributions other than the normal by replacing Φ in the above with other CDFs, which is easily done via generalized linear models (GLMs; McCullagh & Nelder, 1989), in that the inverse of the link function corresponds to a CDF (see DeCarlo, 1998).

1.2. A latent variable formulation

Although Eq. (1) is written in terms of observed variables, the model can also be written in terms of a latent variable, which is how it is motivated in SDT. More specifically, the decision is viewed as being based on an underlying random variable, Ψ, usually conceptualized in psychology as the observer's perception, but Ψ can be conceptualized in a variety of ways, depending on the particular research application. For example, in research on recognition memory, Ψ can be viewed as representing an underlying continuum of familiarity. Note that SDT will be discussed here in terms of recognition memory examples, but the results are general and not tied to any particular application.

The perception on each trial is viewed as being a realization (i.e., $\Psi = \psi$) from a probability distribution, as shown in Fig. 1. The figure shows that an observer chooses a response of "1" if his or her perception is below the first response criterion, a response of "2" if it is between the first and second response criterion, and so on. More explicitly, the decision rule is

$$Y = k \quad \text{if } c_{k-1} < \psi \leq c_k, \qquad (2)$$

where $c_1 < c_2 < \cdots < c_{k-1}$ with $c_0 = -\infty$ and $c_K = \infty$. Next, the observer's perception is related to a stimulus presentation as follows:

$$\Psi = dx + \varepsilon, \qquad (3)$$

where $x = 0$ or 1 for noise or signal, respectively, d is the distance of the mean of the signal distribution from noise (scaled with respect to the square root of $V(\varepsilon)$; see below), and ε represents random variation in the perception; ε has a mean of zero, $E(\varepsilon) = 0$, and variance $V(\varepsilon)$, where E is the expectation operator and V is the variance operator. From the perspective of psychometrics (e.g. Bollen, 1989), Eq. (3) is a *structural equation* that relates the latent construct Ψ to an observed variable X.

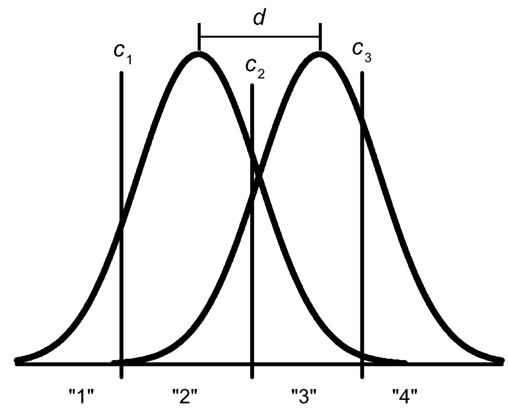

Fig. 1. An illustration of signal detection theory.

Eqs. (2) and (3) are the fundamental equations of SDT, with Eq. (2) representing the decision component and Eq. (3) representing the perceptual component. It follows from Eqs. (2) and (3) that

$$p(Y \leq k \mid x) = p(\Psi \leq c_k \mid x) = p(dx + \varepsilon \leq c_k) = p(\varepsilon \leq c_k - dx).$$

If $\varepsilon \sim N(0, 1)$, then

$$p(\varepsilon \leq c_k - dx) = \Phi(c_k - dx),$$

and so

$$p(Y \leq k \mid x) = \Phi(c_k - dx),$$

which is the equal variance normal SDT model, as shown in Eq. (1). Note that any and all other possible extensions of SDT can be expressed as generalizations or modifications of either the decision rule, Eq. (2), the perceptual component, Eq. (3), or both.

1.3. Conditional means and variances

Deeper insight into the various models discussed here can be obtained by considering the conditional mean and variance of the latent underlying variable, as well as its conditional distribution. For example, it follows directly from Eq. (3) that the conditional mean and variance for the equal variance SDT model are

$$E(\Psi \mid x = 0) = 0, \qquad V(\Psi \mid x = 0) = V(\varepsilon) \qquad (4)$$
$$E(\Psi \mid x = 1) = d, \qquad V(\Psi \mid x = 1) = V(\varepsilon),$$

which shows that the mean of the signal distribution is d, which is the difference between the conditional means (for signal and noise) scaled with respect to $\sqrt{V(\varepsilon)}$, and the variance is $V(\varepsilon)$, which is constant across signal and noise. It also follows from Eq. (3) and the assumption that ε is normal that the conditional distribution of Ψ given X, that is $\Psi|X$, is normal for both signal and noise.

Note that there is an indeterminancy in the scale for models with categorical dependent variables, and so $V(\varepsilon)$ is typically set to unity (for the normal model); for other models, the square root of the variance is used, such as $\sqrt{(\pi^2/3)}$ for the logistic (see DeCarlo, 1998). The scaling affects d and c_k, in that they are both scaled with respect to the square root of $V(\varepsilon)$, and so d and c_k in Eqs. (2), (3) and (4) can be written more accurately as $d/\sqrt{V(\varepsilon)}$ and $c_k/\sqrt{V(\varepsilon)}$. However, $V(\varepsilon)$ is set to unity for (most of) the models considered here (and ε is assumed to be normal), and so the parameters are written more simply as d and c_k, but effects of scaling will be noted where appropriate (e.g., for the varying criteria model discussed below). Also note that $E(\Psi|x = 0)$ is used as the zero point, which defines c_k; another approach is to use the intersection point of the distributions as zero, which gives a different c_k (see DeCarlo, 1998, p. 201).

2. The unequal variance SDT model

2.1. The statistical model

The unequal variance normal SDT model can be written as

$$p(Y \le k \mid x) = \Phi[(c_k - dx)/\sigma^x], \tag{5}$$

(e.g. DeCarlo, 1998, 2003b). The denominator on the right-hand side of Eq. (5), σ^x, allows the standard deviation (and thus the variance) to differ across signal and noise, and is equal to unity for $x = 0$ and σ for $x = 1$. In implementations of the model, it is common to replace σ^x with a term such as $\exp(bx)$, so that the estimate of the standard deviation is strictly positive (see DeCarlo, 2003b). Note that Eq. (5) is not a GLM because it is non-linear in the parameters (because of the term σ^x in the denominator). The model is generally known in statistics and econometrics as a probit model with heteroscedastic error (see DeCarlo, 2003b; Greene, 2003) and can be fit with standard software, such as SAS or SPSS.

2.2. A latent variable formulation

The unequal variance SDT model can also be written as a latent variable model, in the manner of Eq. (3). The model is

$$\Psi = dx + \sigma^x \varepsilon, \tag{6}$$

where $\sigma > 0$; σ is a scale parameter that allows the variance to differ across signal and noise. The decision rule is the same as Eq. (2), from which, with Eq. (6), it follows that

$$\begin{aligned} p(Y \le k \mid x) &= p(\Psi \le c_k \mid x) \\ &= p(dx + \sigma^x \varepsilon \le c_k) = p[\varepsilon \le (c_k - dx)/\sigma^x]. \end{aligned}$$

If $\varepsilon \sim N(0, 1)$, then

$$p[\varepsilon \le (c_k - dx)/\sigma^x] = \Phi[(c_k - dx)/\sigma^x]$$

and so

$$p(Y \le k \mid x) = \Phi[(c_k - dx)/\sigma^x],$$

which is the unequal variance normal SDT model, as given by Eq. (5).

2.3. Conditional means and variances

It follows from Eq. (6) that

$$\begin{aligned} &E(\Psi \mid x = 0) = 0 \qquad V(\Psi \mid x = 0) = V(\varepsilon) \\ &E(\Psi \mid x = 1) = d \qquad V(\Psi \mid x = 1) = \sigma^2 V(\varepsilon). \end{aligned} \tag{7}$$

Eq. (7) shows that the mean for the signal distribution is d, which is the distance scaled with respect to the standard deviation of the noise distribution, $\sqrt{V(\varepsilon)}$, which is set to unity; d in this case is equivalent to the distance measure Δm in the normal version of SDT (see Green & Swets, 1988). There are some well-known issues with respect to the scaling of d in the unequal variance SDT model (that arise because the variance differs across signal and noise); see, for example, Macmillan and Creelman (2005, p. 59). With $V(\varepsilon) = 1$, Eq. (7) shows that σ^2 is simply the variance of the signal distribution.

An important aspect of Eq. (7) is that it shows that the variance for signal is proportional to that for noise by a factor of σ^2, which reflects that σ is a *multiplicative scale parameter*. Thus, the unequal variance SDT model is consistent with signal variance that is greater than (if $\sigma > 1$), equal to (if $\sigma = 1$), or less than (if $\sigma < 1$) the noise variance. It also follows from Eq. (6) and the assumption of normal ε that the conditional distribution of Ψ given X is normal for both signal and noise. Also note that the conditional means and variances in Eq. (7) involve separate parameters, d and σ, and so there is no necessary relation between the mean and variance in the unequal variance normal SDT model. It has sometimes been claimed, for example, that the signal variance must approach the noise variance as d approaches zero, but there is nothing in the SDT model itself that predicts or accounts for this (empirical) result (it is noted below that a mixture SDT model provides an account of this finding). Finally, it should be noted that a fit of the model (for rating responses, or multiple session binary responses) provides estimates of both d and σ, along with the response criteria c_k.

2.4. Empirical versus theoretical models

It is important to recognize that the unequal variance extension of SDT is empirical, in that it simply introduces an additional parameter, σ, to improve fit, and so σ does not have a specific theoretical interpretation; it simply allows the variance to differ. As noted by Green and Swets (1988),

> "The justification for the Gaussian model with unequal variance is, we believe, not to be made on theoretical but rather on practical grounds. It is a simple, convenient way to summarize the empirical data with the addition of a single parameter." (p. 79)

Thus, the unequal variance model does not specify *why* the signal distribution might have a different variance than the noise distribution; it simply allows for it. Note that, as shown by Eq. (4), the signal distribution has a variance of $V(\varepsilon)$ in the equal variance SDT model, and so the question raised by the unequal variance model is why the variance, $V(\varepsilon)$, is inflated or deflated (by σ^2, as shown in Eq. (7)). Because of the lack of a theoretical grounding for σ^2, many studies have in fact simply focused on d and neglected the signal variance, which is often viewed as a nuisance parameter. Note that, with respect to adding additional parameters, a further (empirical) extension is to use distributions with additional (shape) parameters, to allow for non-linearities (e.g., skew and/or kurtosis) in empirical z-ROC curves (cf. DeCarlo, 1998, p. 198).

The theoretical void associated with the unequal variance SDT model has been widely recognized, and so attempts have been made to motivate the model. Two basic motivations for the unequal variance model are considered here. The first allows for the possibility that the SDT parameter d is not constant over trials, but varies. It is shown that a model with varying d leads to a version of the unequal variance SDT model. Further, allowing for varying c_k in this model is shown to not change the unequal variance SDT model in any fundamental way. The second approach considers the influence of a second process (e.g., attention) in detection tasks; the resulting model is a mixture extension of SDT. Similarities and differences between the approaches are noted.

3. SDT models with random coefficients

This section shows that a model with a trial-varying d leads to a model with unequal variances; however, the model is distinct from the traditional unequal variance SDT model. The following section shows that the distinction makes a difference when one considers a model with varying criteria.

3.1. SDT with varying d

3.1.1. Theoretical motivation

It is assumed in the basic SDT model that added strength, as measured by d, is constant over trials. Here the possibility that d varies over trials is considered. The idea can be illustrated in the context of research on recognition memory, where words from a study list (old words) are presented along with new words in a memory test (and the task, for example, is to rate one's confidence that a word is old or new). Wixted (2007) made an argument for the unequal variance SDT model in this situation as follows:

> "The targets can be thought of as lures that have had memory strength added to them by virtue of their appearance on the study list. An equal-variance model would result if each item on the list had the exact same amount of strength added during study. However, if the amount of strength that is added differs across items, which surely must be the case, then both strength and variability would be added, and an unequal variance model would apply." (p. 154)

Put another way, in traditional SDT, the perception of each item is viewed as being a realization from a probability distribution with a mean of d; that is, there is a distribution for each item, each with a mean of d. Note that this implies that, even if the same item was presented repeatedly (i.e., on different trials), the perception would differ. In the varying strength version of the model, the items are viewed as coming from distributions with different d; that is, there is a distribution for each item with a different mean (e.g., $d + \gamma$), and so d differs across items (and trials, given that a different item is presented on each trial), as suggested above by Wixted (2007). Note that a simple motivation of the model in the context of psychophysical research is that the sensitivity (d) of the observer varies over trials.

3.1.2. A latent variable formulation

Eq. (3) with a d that varies over trials can be written as

$$\Psi = (d + \gamma)x + \varepsilon, \tag{8}$$

where γ is a latent variable that reflects variation in d over trials; it is assumed that $\gamma \sim N[0, V(\gamma)]$, $\varepsilon \sim N(0, 1)$, and γ and ε are uncorrelated.[1] The statistical model follows by using Eq. (8) together with the decision rule of Eq. (2).

3.1.3. The statistical model

The statistical model is an extension of Eq. (1),

$$p(Y \le k \mid x, \gamma) = \Phi[c_k - (d + \gamma)x]. \tag{9}$$

Eq. (9) is a probit model with a slope, $d + \gamma$, that varies over trials, and is an example of what is known as a *random coefficient* model in statistics (e.g. Longford, 1993), because the coefficient $d + \gamma$ is a random variable, rather than a fixed constant d. More generally, the model is an example of a generalized linear mixed model (GLMM; see Agresti, 2002; Breslow & Clayton, 1993); GLMMs extend generalized linear models by allowing for random slopes and/or intercepts. From a statistical perspective, the model provides a possible reason for overdispersion (cf. Hinde & Demétrio, 1998). An interesting aspect of Eq. (9) is that it shows that the unequal variance SDT model can be implemented as a multilevel model (for a single subject!), as shown in the Appendix, which offers another perspective on the model.

[1] A reviewer noted that the normal assumption could lead to a negative value of $d + \gamma$, in which case the old words would be less familiar than the (average) new word. The likelihood of this (and whether or not it is a problem) depends on the relative magnitude of $V(\gamma)$ to $V(\varepsilon)$ and the size of d. If $V(\gamma)$ is small relative to d, for example, then negative values will rarely occur. Another possibility is to prevent this by using a distribution such as a truncated normal.

3.1.4. Conditional means and variances

It follows from Eq. (8) that

$$\begin{aligned} E(\Psi \mid x = 0) &= 0 \qquad V(\Psi \mid x = 0) = V(\varepsilon) \\ E(\Psi \mid x = 1) &= d \qquad V(\Psi \mid x = 1) = V(\varepsilon) + V(\gamma). \end{aligned} \tag{10}$$

With the assumption of normal γ and ε, it follows that the conditional distribution of Ψ given X is normal for both signal and noise, as for the normal unequal variance SDT model. Note that there are separate parameters for the conditional means and variances in Eq. (10), as in Eq. (7), and one can obtain estimates of both d and $V(\gamma)$.

Two important clarifications follow from Eq. (10) and the conditional distributions. First, Eq. (10) shows that, if the signal variance is larger than that for noise, then the model of Eq. (9) is indistinguishable, in terms of conditional means, variances, and distributions, from the unequal variance SDT model of Eq. (5). Thus, the varying d assumption provides a basis for the unequal variance SDT model (with larger signal variance). Note that $V(\gamma)$ equals the signal variance minus the noise variance, and so it represents extra variation in the signal distribution compared to the noise distribution, due to varying strength.

Second, Eq. (10) shows that the model also differs in a basic way from the unequal variance SDT model, in that the conditional variance of the signal distribution in Eq. (10) has an additive term, $V(\gamma)$, and not a multiplicative term (σ^2) as in Eq. (7). Given that $V(\gamma) \ge 0$, it follows that the signal variance can be equal to or greater than the noise variance, but not less than it, whereas for the unequal variance SDT model, the signal variance can be greater than, equal to, or less than the noise variance. Thus, a clarification is that the varying d assumption does not lead exactly to the unequal variance SDT model, contrary to prior suggestions, but rather to a closely related, yet distinct, model. Note, for example, that the unequal variance SDT model can deal with z-ROC curves with slopes greater than unity (which indicates that the signal variance is smaller than the noise variance), as have sometimes been found (see Swets, 1986; for examples in remember-know memory studies, see Rotello, Macmillan, & Reeder, 2004), whereas the varying strength model is not consistent with slopes greater than unity (without the introduction of additional parameters, such as a non-zero covariance).

In sum, the conditional means and variances shown above clarify that a model with varying d provides a basis for the unequal variance SDT model. However, the approach only allows for larger signal variance (assuming that ε and γ are uncorrelated), not smaller, whereas unequal variance SDT allows for larger or smaller variance, and so the models are distinct. The next section shows that the distinction has implications when generalizations that allow for varying criteria are considered.

3.2. SDT with varying criteria

Many researchers have considered the possibility that the response criterion (or criteria) varies over trials (e.g. Baird & Noma, 1978; Benjamin, Diaz, & Wee, 2009; Macmillan & Creelman, 2005; Mueller & Weidemann, 2008; Treisman & Williams, 1984; Wickelgreen, 1968). For example, in a well-known article, Wickelgreen (1968) considered SDT models with a normally distributed response criterion. Macmillan and Creelman (2005) refer to this as *inconsistency*, "An inconsistent participant uses a criterion, but changes the location of the cutoff from trial to trial." (p. 46). This section shows that conditional means and variances are again informative about this type of generalization, in that they clarify (and modify) conclusions made in prior research about the effects of varying criteria.

3.2.1. The statistical model

A statistical model that allows the criteria to vary over trials, for either yes/no or rating response tasks, is

$$p(Y \leq k \mid x, \alpha) = \Phi(c_k + \alpha - dx),$$

which generalizes Eq. (1) by including a random intercept, α. Thus, for yes/no responses, the criterion varies randomly over trials, as in Wickelgreen's (1968) model, whereas for rating responses, the criteria randomly shift up or down (together) over trials.

3.2.2. A latent variable formulation

Note that, for a simple binary response model, including a random intercept means that the decision rule of Eq. (2) is changed to

$$Y = 1 \quad \text{if } -\infty < \psi \leq c_1 + \alpha;$$

else $Y = 2$. The random variable α reflects the amount that the criterion location varies over trials, with $\alpha \sim N[0, V(\alpha)]$. Note that the above can be rewritten to allow the underlying distributions to shift, rather than the criterion:

$$Y = 1 \quad \text{if } -\infty < \psi + \alpha \leq c_1;$$

else $Y = 2$. The above simply reflects the fact that a model where the criterion varies is equivalent to a model where the means of the distributions vary (together).

Note that, for ordinal responses, one should not simply allow each criterion, c_k, to be (independently) random, because the assumption of strict ordering of the criteria can then be violated, as has been recognized in statistics (e.g., see Fahrmeir & Tutz, 2001) and in psychology (e.g. Rosner & Kochanski, 2009). A simple alternative is to allow the criteria to shift together by the same amount, which can be done by generalizing Eq. (2) as

$$Y = k \quad \text{if } c_{k-1} + \alpha < \psi \leq c_k + \alpha.$$

The statistical model shown above can then be derived by using this assumption together with Eq. (3). To show the effects on the conditional means and variances; however, it is useful to recognize that once again a model with shifting criteria is equivalent to a model with shifting distributions, and so the above can be rewritten as

$$Y = k \quad \text{if } c_{k-1} < \psi + \alpha \leq c_k.$$

The structural model is

$$\Psi' = \Psi + \alpha = dx + \varepsilon + \alpha. \tag{11}$$

It follows that

$$p(Y \leq k \mid x, \alpha) = \Phi(c_k + \alpha - dx),$$

which is the statistical model as given above (the derivation gives a minus sign for α, which is irrelevant for symmetric distributions such as the normal).

3.2.3. Conditional means and variances

It follows from Eq. (11) that

$$\begin{aligned} &E(\Psi' \mid x = 0) = 0 \qquad V(\Psi' \mid x = 0) = V(\varepsilon) + V(\alpha) \\ &E(\Psi' \mid x = 1) = d \qquad V(\Psi' \mid x = 1) = V(\varepsilon) + V(\alpha), \end{aligned} \tag{12}$$

which shows that the effect of variability in the location of the criteria is simply to increase the noise and signal variance additively by $V(\alpha)$. With the assumption that $\alpha \sim N[0, V(\alpha)]$, it follows that the signal and noise distributions are normally distributed, as in the equal variance normal SDT model. Eq. (12) shows that the criteria variance, $V(\alpha)$, is confounded with the perceptual variance, $V(\varepsilon)$. More importantly, Eq. (12) shows that allowing for randomly shifting criteria does not change the equal variance SDT model in any fundamental way; it simply adds to the signal and noise variance. Thus, the equal variance SDT model is consistent with a model where the criteria shift randomly over trials; the only practical consequence is that d is underestimated, because the scaling sets $V(\varepsilon) + V(\alpha) = 1$ instead of $V(\varepsilon) = 1$, and so $d/\sqrt{[V(\varepsilon)+V(\alpha)]}$ is estimated, and not $d/\sqrt{V(\varepsilon)}$; c_k is similarly affected. Underestimation of d means that the obtained estimate of discrimination is conservative.

3.3. SDT with varying d and varying criteria

3.3.1. A random slope and intercept

Allowing for both varying d and shifting criteria gives a model with a random slope and intercept,

$$p(Y \leq k \mid x, \alpha, \gamma) = \Phi[c_k + \alpha - (d + \gamma)x]. \tag{13}$$

The conditional means and variances are

$$\begin{aligned} &E(\Psi' \mid x = 0) = 0 \qquad V(\Psi' \mid x = 0) = V(\varepsilon) + V(\alpha) \\ &E(\Psi' \mid x = 1) = d \qquad V(\Psi' \mid x = 1) = V(\varepsilon) + V(\alpha) + V(\gamma). \end{aligned} \tag{14}$$

It also follows that the conditional distribution is normal for both signal and noise. Thus, the conditional means, variances, and distributions show that the varying d/varying criteria SDT model is not distinguishable from the unequal variance normal SDT model (if the signal variance is larger than the noise variance). Put another way, the traditional unequal variance SDT model is consistent with a model where the response criteria and discrimination strength vary randomly over trials.

Note that the scale is set by setting the conditional variance of the noise distribution to unity, and so $V(\varepsilon) + V(\alpha) = 1$ (and so d is again underestimated). It follows that the signal variance is $1 + V(\gamma)$, and so the slope of the z-ROC curve, which is the ratio of the noise to signal standard deviations, is

$$\frac{1}{\sqrt{1 + V(\gamma)}}. \tag{15}$$

Eq. (15) shows that the slope of the z-ROC curve depends solely on $V(\gamma)$; it follows that the z-ROC slope provides an estimate of $V(\gamma)$, just as it provides an estimate of σ^2 in the unequal variance SDT model (given that $V(\gamma) = \sigma^2 - 1$).[2]

The above conclusion differs somewhat from that reached in other research (e.g. Benjamin et al., 2009; Wickelgreen, 1968). For example, Wickelgreen (1968) noted that "Notice that when criterion variance is considered, it is clear that the slope of the operating characteristic does *not* provide a measure of the ratio of the standard deviations of the two s-distributions, as has previously been assumed." (p. 107). Here it is shown that this conclusion occurred because the empirical unequal variance SDT model was generalized; it does not hold for the model of Eq. (13).

In particular, if a random criterion (or shifted criteria, as above) is included in the empirical unequal variance SDT model, then it follows that the conditional means and variances of Eq. (7) become

$$\begin{aligned} &E(\Psi' \mid x = 0) = 0 \qquad V(\Psi' \mid x = 0) = V(\varepsilon) + V(\alpha) \\ &E(\Psi' \mid x = 1) = d \qquad V(\Psi' \mid x = 1) = \sigma^2 V(\varepsilon) + V(\alpha); \end{aligned} \tag{16}$$

(compare Eq. (16) to Eq. (14)). It follows that the slope of the z-ROC curve is the square root of the ratio of $V(\varepsilon) + V(\alpha)$ to $\sigma^2 V(\varepsilon) + V(\alpha)$, as also shown, for example, by Eq. (5) of Wickelgreen (1968) and by Eq. (2) of Benjamin et al. (2009). As before, the variance of the noise distribution is set to unity, and so $V(\varepsilon) + V(\alpha) = 1$.

[2] It can be shown that this will be the case even if the response criteria have a correlated structure, such as first-order autocorrelation, $\alpha_t = \rho\alpha_{t-1} + \upsilon_t$.

Noting that $V(\varepsilon) = 1 - V(\alpha)$, it follows from Eq. (16) that the slope of the z-ROC curve is

$$\frac{1}{\sqrt{\sigma^2 + (1-\sigma^2)V(\alpha)}}. \tag{17}$$

Eq. (17) shows that there is a relation between the slope of the z-ROC curve and variability in the criteria, $V(\alpha)$. Specifically, Eq. (17) shows that an increase in $V(\alpha)$ gives an increase in the slope if $\sigma^2 > 1$ and a decrease in the slope if $\sigma^2 < 1$, as was also noted by Benjamin et al. (2009). Eq. (17) also illustrates Wickelgreen's point that the slope of the z-ROC curve no longer simply provides an estimate of the variance, σ^2, because it also depends on $V(\alpha)$, in contrast to Eq. (15).

In sum, the conditional expectations and variances again provide important clarifications. In particular, they show that, if the response criteria vary over trials, and/or if discrimination d varies over trials, then the traditional unequal variance SDT model does not necessarily change in any fundamental way (i.e., apart from underestimation of d). Apparent changes discussed in other articles, such as a relation between the z-ROC slope and criterion variability, arose because the (empirical) unequal variance extension of SDT was generalized; the relation does not appear if the varying strength SDT model is instead generalized (this is also the case if random criteria are included in the mixture SDT model discussed below). Thus, the theory and formalization of the model are crucial and can heavily influence the conclusions. The above derivations provide a framework within which to examine the implications and utility of further extensions of the models that might be considered.

4. The mixture SDT model

Another approach to the unequal variance SDT model is to consider the possibility that the "unequal variances" might result from a mixture process involving distributions with equal variances. This section shows the formal relation of the mixture SDT model to the unequal variance SDT model, which clarifies similarities and differences between the models.

4.1. Theoretical motivation

The mixture SDT model is motivated by considering additional processes that might operate in signal detection experiments. For example, many researchers have recognized that attention likely has an effect on perception, as was noted by Macmillan and Creelman (2005):

> An inattentive observer dozes off, or at least drifts into reverie, on some proportion of trials; because failing to respond is usually discouraged, this leads to an unknown number of $d' = 0$ trials, ones on which the observer responds despite not having paid attention, mixed in with the others (p. 46)

This is exactly the idea that underlies the mixture SDT model, as discussed in DeCarlo (2000, 2002); also see DeCarlo (2003a, 2007, 2008). The basic idea is that, on some proportion of trials, the observer does not attend to the signal, and so the distribution is not shifted (as noted below and in the Appendix, a more general version of the model allows for partial attention).

4.2. The statistical model

From a statistical perspective, the model is a mixture of cumulative probit models (or more generally, a mixture of GLMs),

$$p(Y \le k \mid x) = \lambda\Phi(c_k - dx) + (1-\lambda)\Phi(c_k - d^* x) \tag{18}$$

where λ is a mixing parameter. From the perspective of SDT, the above is a mixture of equal variance SDT models, where λ can be interpreted as reflecting the effects of attention (e.g., the proportion of items that were attended to) and d^* is the location of a partially attended distribution. Some notes on this version of the model are given in the Appendix. Considered here is a simple yet useful version where d^* is restricted to be zero,

$$p(Y \le k \mid x) = \lambda\Phi(c_k - dx) + (1-\lambda)\Phi(c_k). \tag{19}$$

Eq. (19) has a simple interpretation in terms of each item either being attended to or not attended to, exactly as in Macmillan and Creelman's description; it has previously been shown that the restricted model is useful for recognition memory data (DeCarlo, 2002). Note that mixture SDT maintains the assumption that a stimulus presentation shifts the location of a normal underlying distribution by a constant d with no effect on the variance, and so d has a simple interpretation (in terms of memory strength, for example) and scaling problems do not arise (because the unmixed distributions all have equal variances). Mixture SDT also introduces a mixing parameter, λ, which can be interpreted as a measure of attention (there are of course other possible interpretations).

4.3. A latent variable formulation

The mixture SDT model of Eq. (19) follows directly from the theory as described above, as shown earlier (DeCarlo, 2002). Another (statistical) perspective on the model, given here, is that it is a type of random coefficient model (with a discrete random slope). Expressing the model in this way allows one to derive basic results for the conditional means and variances, as done for the other models presented above; it also shows how the model can be implemented in software for latent class analysis, as shown in the Appendix.

In particular, a structural equation for mixture SDT is

$$\Psi = (\delta d)x + \varepsilon, \tag{20}$$

where δ is a latent binary variable, and in particular, δ is a Bernoulli variable that takes on values of 0 and 1, with the probability of 1 given by λ; that is, $\delta \sim B(\lambda)$, and so $E(\delta) = \lambda$ and $V(\delta) = \lambda(1-\lambda)$. The latent binary variable δ can be interpreted as reflecting whether or not a signal was attended to. Eq. (20) shows that attention in the mixture SDT model is viewed as having a moderating effect on perception, as reflected by the multiplicative interaction of δ and d.

It follows from Eq. (20), along with the decision rule of Eq. (2), that

$$\begin{aligned} p(Y \le k \mid x, \delta) &= p(\Psi \le c_k \mid x, \delta) \\ &= p(\delta dx + \varepsilon \le c_k) = p(\varepsilon \le c_k - \delta dx). \end{aligned}$$

If $\varepsilon \sim N(0, 1)$, then

$$p(\varepsilon \le c_k - \delta dx) = \Phi(c_k - \delta dx),$$

and so

$$p(Y \le k \mid x, \delta) = \Phi(c_k - \delta dx). \tag{21}$$

Eq. (21) shows the form in which the model can be fit with software for latent class analysis, such as LEM (Vermunt, 1997) or Latent Gold (Vermunt & Magidson, 2007); sample programs are given in the Appendix.

Note that Eq. (21) is conditional on both X and δ, whereas Eq. (19) is conditional on X alone. Taking the expectation over δ gives a model conditional on X,

$$\begin{aligned} p(Y \le k \mid x) &= E_\delta[p(Y \le k \mid x, \delta)] = E_\delta[\Phi(c_k - \delta dx)] \\ &= \Sigma_\delta[\Phi(c_k - \delta dx)]p(\delta) \\ &= \lambda\Phi(c_k - dx) + (1-\lambda)\Phi(c_k), \end{aligned}$$

given that $\delta \sim B(\lambda)$, and so $p(\delta = 1) = \lambda$. The above gives the model as shown in Eq. (19).

Table 1
Parameter estimates for manipulation of repetitions and attention at study, Dunn (2009).

	Unequal variance SDT			
	Focused attention		Divided attention	
Repetitions	d	σ^2	d	σ^2
1	1.36	1.58	1.02	1.72
2	1.82	1.59	1.36	1.38
4	2.38	1.90	1.72	1.45
	Mixture SDT			
	Focused attention		Divided attention	
Repetitions	d	λ	d	λ
1	1.60	.81	1.58	.62
2	1.91	.90	1.52	.87
4	2.31	.93	1.80	.91

4.4. Conditional means and variances

It follows from Eq. (20) and the assumption that $\delta \sim B(\lambda)$ with $E(\delta) = \lambda$ and $V(\delta) = \lambda(1 - \lambda)$ that

$$E(\Psi \mid x = 0) = 0 \qquad V(\Psi \mid x = 0) = V(\varepsilon) \qquad (22)$$
$$E(\Psi \mid x = 1) = \lambda d \qquad V(\Psi \mid x = 1) = V(\varepsilon) + d^2\lambda(1 - \lambda).$$

Eq. (22) shows that, for signal presentations, the conditional mean depends on the proportion of trials on which the signal was attended to. For example, if the signal was attended to 60% of the time, then the mixed distribution has a location of .6*d*, which simply reflects the fact that it is located at *d* 60% of the time and at zero 40% of the time.

It also follows from Eq. (22) that, as *d* approaches zero, the signal variance will approach the noise variance, because $d^2\lambda(1 - \lambda)$ approaches zero. Thus, the mixture SDT model is consistent with the finding of *z*-ROC curves that approach unity as *d* approaches zero. Also note that Eq. (22) shows that the signal variance reflects changes in both *d* and/or λ, because of the term $d^2\lambda(1 - \lambda)$. That is, although separate estimates of *d* and λ are provided by fits of the mixture model, Eq. (22) shows that they are confounded in the variance estimates, σ^2 and $V(\gamma)$, in the unequal variance and varying strength models, and so the variance estimates are not necessarily informative. This can be illustrated by putting numerical values into Eq. (22), but real data that nicely illustrate the point are available.

4.5. An example: Does repetition increase or decrease the variance?

Table 1 shows parameter estimates obtained for fits of the unequal variance and mixture SDT models to data from an experiment by Dunn (2009).[3] The experiment manipulated, within subjects, the number of times each word was presented during study (once, twice, or four times) and whether the word received focused or divided attention. For focused attention, each word was presented for 2 s by itself, whereas, for divided attention, each word was flanked by a digit on each side (for the first 200 ms; the digits differed in value and in physical size) and, after the 2 s word presentation was completed, the participant had to indicate which side the digit with a larger numerical value was on, or which side the physically larger digit was on.

With respect to fit, both the unequal variance and mixture SDT models describe the data; the goodness of fit likelihood ratio (LR) statistic for the mixture SDT model is 13.09 with 18 degrees of freedom (*df*) and $p = .79$ whereas for the unequal variance model the LR is 15.03 with 18 *df* and $p = .66$. Of interest here are the parameter estimates, which illustrate aspects of Eq. (22).

The top part of Table 1 shows results for the unequal variance SDT model. It is apparent that *d* increases with the number of repetitions, both for full and divided attention words. With respect to the variance, the variance for full attention words tends to increase with the number of repetitions, whereas the variance for divided attention words tends to decrease. This raises questions as to why the variance increases with repetitions for full attention but decreases for divided attention. Note that this also raises questions for the varying strength model, given that the only difference is that $V(\gamma) = \sigma^2 - 1$. Table 1 also shows that, for one repetition, the variance is larger for divided attention than for full attention, but the opposite is true for two and four repetitions. Thus, results for both the unequal variance and varying strength SDT models show a complex pattern, in terms of changes in the variance across repetitions and across full/divided attention conditions.

The lower portion of Table 1 shows results for fits of the mixture SDT model. Once again, *d* increases with the number of repetitions for both full and divided attention words. Of particular interest is that the estimates of λ increase with the number of repetitions for both full and divided attention. This is consistent with a simple interpretation of λ in terms of attention — the proportion of attended words increases when words are presented more than once. Second, note that, for each repetition number, λ is smaller in the divided attention condition than in the full attention condition, and so using a task that divides attention at study yields smaller values of λ, which again supports the view of λ as a measure of attention. Thus, estimates of λ are ordered in a consistent and simple manner across both the rows and columns of Table 1, whereas estimates of σ^2 and $V(\gamma)$ are not. The mixture model unifies the results and offers a simple summary: repetition, for both full and divided words, increased the proportion of words that were attended to, as reflected by λ, and the full/divided attention manipulation indeed affected attention, in that λ was smaller for divided attention words.

The example illustrates that results for the mixture SDT model can differ considerably from those obtained for the unequal variance or varying strength SDT models. In this case, the mixture model results suggest that the apparent differences arose because λ was closer to .5 in the divided attention condition, and so an increase in λ with repetitions (from .62 to .91) gives a fairly large decrease in $\lambda(1 - \lambda)$, which makes the variance appear to be smaller for a fit of the unequal variance SDT model (even with an increase in *d*; see Eq. (22)). On the other hand, this did not happen in the focused attention condition because λ was larger and so an increase in λ with repetitions (from .81 to .93) gives a smaller decrease in $\lambda(1-\lambda)$ that is not sufficient to counteract the effect of an increase in *d*, and so the variance appears to be larger. This shows that it is informative to fit the mixture SDT model, in addition to the traditional unequal variance model.

4.6. Conditional distributions

Another basic way in which the mixture model differs from the unequal variance and varying strength models is that the signal distribution is not necessarily normal, but rather is a mixture of normals (for $0 > \lambda < 1$). A number of studies have provided evidence of non-normality, which appear as non-linearities in *z*-ROC curves, that is consistent with simple mixing.

Recognition memory. Shimamura and Wickens (2009) recently presented an informative example of non-normality in recognition memory (data for young subjects from Experiment 5 of Glanzer, Hilford, & Kim, 2004). Table 2 presents a table similar to Shimamura and Wickens' Table 1. The table shows likelihood ratio (*LR*) goodness of fit statistics (denoted as G^2 in Shimamura and

[3] The author thanks John Dunn for making his data available.

Table 2
Fit statistics for item recognition, Experiment 5, young subjects, Glanzer et al. (2004).

Model	*LR*	*df*	*p*	AIC	BIC
Unequal variance SDT	13.859	3	.003	12,084	12,128
Dual process	21.010	3	<.001	12,091	12,135
Restricted mixture SDT	20.680	3	<.001	12,091	12,135
Ex-Gaussian SDT	9.280	2	.010	12,081	12,132
Mixture SDT	5.343	2	.069	12,077	12,127

Note: the results are identical to those shown in Table 1 of Shimamura and Wickens (2009), with the correction that the *LR* statistic for the Dual Process model is 21.010 (and not 21.610), and versions of the information criteria based on the log likelihood are used.

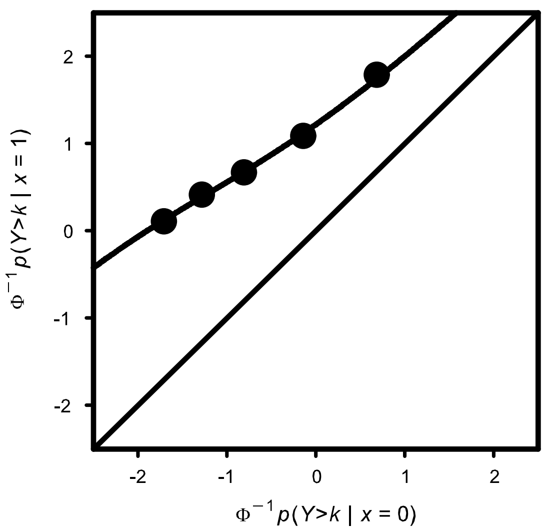

Fig. 2. Data for young subjects, Experiment 5, Glanzer et al. (2004), recognition condition. The solid line is a z-ROC curve obtained from a fit of mixture SDT.

Wickens' table) and values of the information criteria BIC and AIC (versions based on the log-likelihood, as used in previous studies, are shown here; Shimamura and Wickens used versions based on G^2; which version is used does not make any difference; also note that the value of AIC for the Dual Process Model in Shimamura and Wickens' Table 1 should be 35.01, and not 25.01). The models (and results) shown in the table are the same as those considered by Shimamura and Wickens, except that a mixture SDT model with non-zero values of d^* (Eq. (18)) is included at the bottom of the table.

With respect to absolute fit (the *LR* statistic), the table shows that all of the models are rejected (at the 0.05 level), except for the mixture model of Eq. (18). With respect to relative fit, the information criteria BIC and AIC are both smallest for the mixture SDT model compared to any of the other models. A plot of the z-ROC curve helps to clarify why this is so. As shown in Fig. 2, the data show some non-linearity, and in particular the z-ROC curve has a slight "kink", in that it bends down and then up, which is a characteristic of z-ROC curves for mixture SDT, as shown in Figure 3 of DeCarlo (2002). Thus, the unequal variance SDT model is rejected in Table 2 by the LR test because it is inconsistent with the non-linearity shown in Fig. 2. Similarly, the ex-Gaussian model is rejected because it cannot account for the reverse bend shown in the figure (the model gives a z-ROC curve that bends upwards). The mixture SDT model, on the other hand, describes the data, as shown in Fig. 2 and Table 2. This shows that mixture SDT can account for distributions with more complex shapes than simply skew (see the mirror effect and process dissociation section below). Note that the underlying distributions in mixture SDT are all normal; the apparent non-normality arises from mixing over trials.

Source discrimination. Another example of non-normality comes from source discrimination studies, which have consistently found z-ROC curves that bend upwards and so have a U-shape (see DeCarlo, 2003a; also see Slotnick & Dodson, 2005). The curvature is not consistent with the unequal variance and varying strength SDT models, because they predict linear z-ROC curves. Mixture SDT, on the other hand, accounts for the non-linearity by using the same mechanism as used in simple recognition, namely mixing caused by a lack of attention, in this case to the different sources (for details, see DeCarlo, 2003a). Thus, the mixture approach can account for different types of non-linearity in z-ROC curves, such as a reverse bend for recognition (Fig. 2) versus upwards curvature for discrimination.

Mirror effect and process dissociation. Non-linearities in z-ROC curves have also been found in studies of the mirror effect and the process dissociation procedure (DeCarlo, 2007, 2008), in the form of a reverse bend, as in Fig. 2. The unequal variance and varying strength SDT models, in their current forms, are not consistent with these results. Mixture SDT again provides a mechanism for the non-normality, namely mixing due to a lack of attention, and so the approach unifies results found across recognition, source discrimination, the mirror effect, and process dissociation experiments, all with the same simple mechanism.

4.7. Similarities and differences

As shown above, the varying strength and mixture SDT models are similar, statistically, in that they are both types of random coefficient models. Note, however, that $d+\gamma$ in the varying strength model has a continuous distribution (normal), whereas δd in the mixture model has a discrete distribution (Bernoulli). Note that one can also formulate a non-parametric version of the varying strength model by replacing the continuous additive random variable (γ) with a discrete additive random variable (with more than two classes), as done in non-parametric generalizations of random coefficient models (e.g., see Vermunt & van Dijk, 2001).

Here it is noted that the mixture and varying strength models also differ mathematically and conceptually, over and above the discrete/continuous distinction. For example, the mixture model uses a multiplicative term, and not an additive term, which reflects the conceptualization in terms of a moderating effect of attention (and so adding more latent classes does not simply turn the mixture model into the varying strength model; note that prior research has also shown that two classes appear to be sufficient). A basic difference is that, in the mixture model, a lack of attention (or partial attention) can only reduce d, whereas, in the varying strength model, γ can increase or decrease d. Another difference is that, in the mixture model, when strength is added, a constant amount d is added, as in traditional equal variance SDT, whereas this is not the case in the varying strength model (where a random amount is added). In short, the models are related (and could both be embedded in a larger model) but are also distinct, conceptually and mathematically; as shown here, the devil is in the detail.

5. Conclusions

An understanding of the conditional means, variances, and distributions associated with an SDT model is essential in order to fully understand the model and its implications. The approach helps to clarify what the unequal variance SDT model does and does not do, for example, and what various generalizations do. It is shown that framing the models in terms of the decision model of Eq. (2) and perceptual model of Eq. (3) clarifies important details, both conceptual and statistical. The basic results provided here can be built upon for other extensions of SDT that have been, or will be, proposed.

Appendix

A.1. A general mixture SDT model

In the version of the mixture model presented in DeCarlo (2002, Equation 1), the unattended distribution does not necessarily have the same location as the noise distribution, but rather the model allows for a partially attended distribution that has a location other than zero (i.e., $d^* \neq 0$). Some basic results for this version of the model are derived here; results for the restricted version with $d^* = 0$ are presented in the text.

The statistical model is given by Eq. (18). The decision rule is the same as Eq. (2), whereas the structural equation is

$$\Psi = [\delta(d - d^*) + d^*]x + \varepsilon,$$

and so the signal distribution is located at d^* for $\delta = 0$ and at d for $\delta = 1$. It follows that

$$E(\Psi \mid x = 0) = 0$$
$$E(\Psi \mid x = 1) = \lambda(d - d^*) + d^* = \lambda d + (1 - \lambda)d^*,$$

which reduces to Eq. (22) when $d^* = 0$. The above shows that allowing for partial attention, $d^* > 0$, results in the mixed distribution having a mean that is larger than λd by $(1 - \lambda)d^*$.

With respect to the conditional variance, it follows that

$$V(\Psi \mid x = 0) = V(\varepsilon)$$
$$V(\Psi \mid x = 1) = V(\varepsilon) + (d - d^*)^2\lambda(1 - \lambda),$$

which shows that the variance depends on the value of d^*, and in particular, the variance is smaller when $d^* > 0$. Note that the above reduces to Eq. (22) when $d^* = 0$.

The statistical model follows directly from the decision rule and the structural equation given above. Note that the model can be written as

$$p(Y \leq k \mid x, \delta) = \Phi[c_k - d^*x - (d - d^*)\delta x],$$

which shows that the model can be fit by simply including x and the interaction δx as predictors, with δ specified as zero/one.

A.2. A latent gold program for the mixture SDT model

The mixture model of Eq. (21) can be specified in Latent Gold as follows.

```
model
title "Swets, Tanner, Birdsall 1961 data - mixture SDT
model";
options
   bayes
      categorical = 0 variances = 0 latent = 0 poisson = 0 ;
      standarderrors = standard;
output parameters profile bvr identification;
variables
   dependent y probit;
   independent x;
   latent
      delta ordinal 2 score = (0 1);
equations
   delta <- 1 ;
   y <- 1 + x delta;
end model
```

Note that delta is declared as "ordinal" (it could also be declared as nominal) so that one can control how scores are assigned to it. To fit the more general mixture SDT model with non-zero d^*, as discussed above, the model syntax simply becomes y <- 1 + x + x delta (see the last equation given above). An LEM program for mixture SDT is available at the author's website.

A.3. Fitting the varying strength SDT model as a GLMM

The unequal variance SDT model can be fit with widely available software such as SPSS or SAS; however, the varying strength model suggests other interesting ways to implement the model. For example, it follows from Eq. (9) that the model can be fit as a multilevel model with a random slope, using software for multilevel modeling. The results will be equivalent to those obtained for a fit of the unequal variance SDT model, when the signal variance is larger than that of noise.

Note that the models discussed here are for the analysis of individual data, with coefficients that are random over trials, whereas in the more typical multilevel application the coefficients are random over subjects. Thus, rather than treating trials as Level 1 and subjects as Level 2, as in the usual multilevel approach (for repeated response data), the trials are treated here as Level 2 and the response as Level 1, with Level 1 consisting of one observation (e.g., there is only one response per trial, though there could be multiple responses per trial). In other words, the trial is the "between" component and the single response per trial is the "within" component, and so $V(\gamma)$ is the "between" variance (i.e., the variance of d over trials) and $V(\varepsilon)$ is the "within" variance (i.e., the variance of the perception on each trial). Note that, even with only one response at Level 1 (the within component), the model is identified for a categorical response variable, because the within variance, $V(\varepsilon)$, is set to unity (for the normal model). The variance of the signal distribution, σ^2, is then equal to $V(\gamma) + 1$ for the random slope model.

It follows that the random slope SDT model of Eq. (9) can be fit using software for multilevel models. An example using Mplus (see Muthén & Muthén, 1998–2007) is as follows.

```
TITLE: Swets, Tanner, Birdsall 1961 data - unequal variance
model via multilevel;
DATA: FILE IS C:\Documents and Settings\My Documents\
swets.dat;
VARIABLE: NAMES ARE trial x y;
   CATEGORICAL = y;
   WITHIN = x;
   CLUSTER = trial;
ANALYSIS: type = twolevel random; link = probit;
estimator = ml;
MODEL:
   %within%
   s|y ON x;
   %between%
   s ON ;
   y@0;
OUTPUT: tech1;
```

where the CATEGORICAL statement identifies y as an ordinal response variable; the WITHIN statement identifies x as a within independent variable (coded as 0, 1); CLUSTER is used to declare "trial" (which is simply the trial number) as a between variable. The MODEL statement specifies a probit model in the "%within%" component of y ON x, with a random slope s. The "%between%" component specifies a regression of the slope s ON an intercept (included by default); the following line restricts the variance of y to zero, so that an estimate of $V(\gamma)$ is obtained (see the technical appendix of the Mplus manual). The results are identical to those obtained for a fit of the unequal variance SDT model.

A Latent Gold program to fit the varying strength model (and so the unequal variance SDT model), using a slightly different approach that is again suggested by Eq. (9), is as follows.

```
model
title 'Swets, Tanner, Birdsall 1961 data - unequal
variance SDT model via LG';
options
   bayes
   categorical = 0 variances = 0 latent = 0 poisson = 0;
   standarderrors = standard;
   output parameters profile bvr classification;
variables
   dependent y probit;
   independent x;
   latent
      gamma continuous;
equations
   gamma ;
   y <- 1 + x + (1)gamma x;
end model
```

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